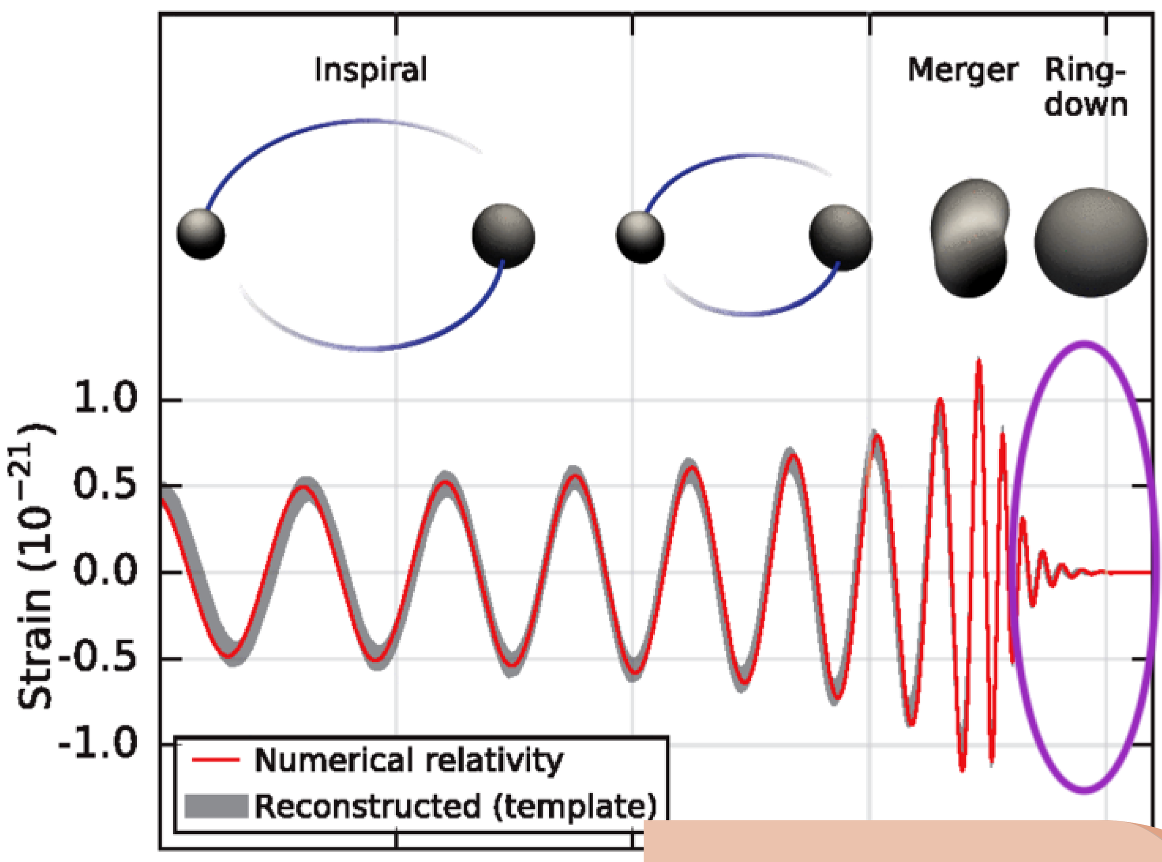
Quantum Integrability as a new method for exact results on black holes' observables and N=2 supersymmetric gauge theories

Davide Fioravanti, Daniele Gregori
University of Bologna / INFN, Italy
daniele.gregori6@unibo.it

From gauge to gravity and back

In the last two years, a surprising connection between $\mathcal{N}=2$ supersymmetric (SUSY) $SU(2)$ gauge theories (Nekrasov-Shatashvili deformed) and black holes (BHs) perturbation theory has emerged [**arXiv:2006.06111**, 2105.04245, 2105.04483, arXiv:2109.09804].

Indeed, G. Aminov, A. Grassi and Y. Hatsuda first found that quantization conditions on the gauge periods a, a_D allow to compute the quasinormal modes (QNMs) ω_n spectrum of black holes from gauge theory methods. QNMs are responsible the damped oscillations appearing, for example, in the ringdown phase of BHs merging.



Unexpected application of extended SUSY to BHs "real physics"!

The importance of this result is manifold.

1. It constitutes a novel analytic characterization of QNMs, for which previously very few analytical and mainly numerical results were known [arXiv:2006.06111].
2. In the increasingly growing outflow of research on this topic, it has already allowed to find new results for the BHs theory, such as:
   (a) an isospectral simpler equation to the perturbation ODE [arXiv:2007.07906];
   (b) improved theoretical proofs of BHs stability [arXiv:2105.13329];
   (c) more accurate computations of observable quantities such as Love numbers, describing tidal deformations [arXiv:2105.04483];
   (d) an simpler interpretation of Chandrasekhar transormation as exchange of gauge mass parameters [arXiv:2111.05857];
   (e) precise determination of the conditions of invariance under (Couch-Torrence) transormations which exchange inner horizon and null infinity [arXiv:2203.14900].
3. It constitutes an unexpected application of Supersymmetry, which was originally thought to describe elementary particles, but has not yet been found by experimentalists.

Integrability and N=2 SUSY gauge theory

Using the approach of the ODE/IM correspondence (see below) we connected basic integrability functions - the Baxter's Q, T and Y functions - to the gauge theory periods a, a_D (from which the prepotential can be obtained). We proved the relations:

$$Q(\theta,P) = \exp 2\pi i a_D(\hbar,u,\Lambda)$$
$$T(\theta,P) = 2\cos 2\pi a(\hbar,u,\Lambda)\,. \tag{1}$$

under the parameters correspondence

$$\frac{\hbar}{\Lambda} = \frac{\epsilon_1}{\Lambda} = e^{-\theta} \qquad \frac{u}{\Lambda^2} = \frac{1}{2}P^2 e^{-2\theta}\,, \tag{2}$$

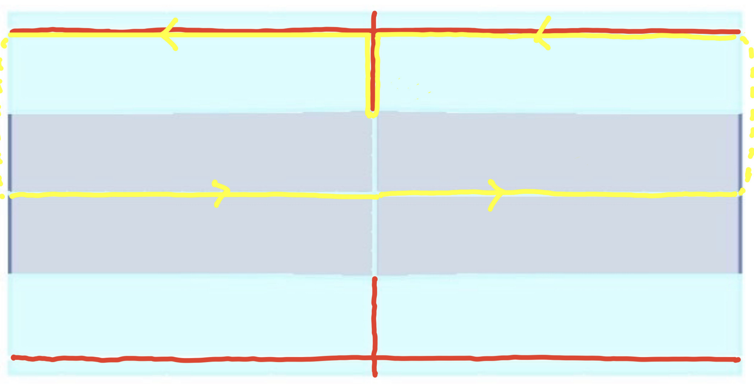


Figure 1: y-integration contour (yellow), branch cuts (red). The integral on the real line gives $\ln Q$, around the branch cuts a_D.

here for the self-dual Liouville IM and $SU(2)$ $N_f=0$ gauge theory [**arXiv:1908.08030**], but also similar ones for the Perturbed Hairpin IM and $SU(2)$ $N_f=1,2$ [**Fioravanti,Gregori,Shu-to appear**] and $SU(3)$ $N_f=0$ gauge theories [**arXiv:1909.11100**].

This fundamental identification allowed us to find several new interesting results for both sides of this new kind of Integrability/Gauge correspondence, for instance:

1. an exact non linear integral equation (Thermodynamic Bethe Ansatz, TBA) for the gauge periods;
2. an interpretation of the integrability functional relations as new exact R-symmetry relations for the periods;
3. new formulas for the local integrals of motion in terms of gauge periods.

For instance, the TQ relation

$$T(\theta,u) = \frac{Q(\theta - i\pi/2, -u) + Q(\theta + i\pi/2, -u)}{Q(\theta,u)} \tag{3}$$

turns out to be a new exact R-symmetry relation for the periods, reducing to the known asymptotic ones in the limit $\theta \to \infty$ (for the gauge periods expansion modes $a^{(n)}, a_D^{(n)}$)

$$a_D^{(n)}(-u) = i(-1)^n\left[-\operatorname{sgn}(\Im u)\, a_D^{(n)}(u) + a^{(n)}(u)\right]. \tag{4}$$

ODE/IM correspondence

In this classic approach to integrability [**arXiv:9812211,9812247,9906219**], the Q function is typically the wronskian of the regular solutions at different singular points

$$Q = W[\psi_+, \psi_-] \qquad \psi_\pm(y) \to 0 \quad y \to \pm\infty \tag{5}$$

of some ODE, like (for self-dual Liouville IM or $SU(2)$ $N_f=0$ gauge theory) [**arXiv:1908.08030**]:

$$-\frac{d^2}{dy^2}\psi + \left[e^{2\theta}(e^y + e^{-y}) + P^2\right]\psi = 0\,. \tag{6}$$

This innovates ODE/IM cor. itself because such ODEs have 2 irregular singularities rather than just 1 as usual. One can derive also T, Y functions as well as the functional and integral equations they satisfy.

An elegant approach to integrability which allows to apply it to very different physical theories!

Integrability and quasinormal modes of black holes

The same ODE is obtained for the (scalar) perturbation of a BH (here D3 brane), under the change of parameters [arXiv:2105.04245]

$$r = Le^{\frac{y}{2}} \qquad \omega L = -2ie^{\theta} \qquad P = \frac{1}{2}(l+2)\,, \tag{7}$$

QNMs are defined precisely as the complex frequencies of expansion of a BH perturbation

$$\Phi(t,x) = \sum_n e^{i\omega_n t}\operatorname{Res}\left(\frac{1}{W(\omega)}\right)\Bigg|_{\omega_n}\int_{-\infty}^{\infty}\Psi_-(\omega_n, x_<)\Psi_+(\omega_n, x_>)\mathcal{I}(\omega_n, x')\,dx'\,. \tag{8}$$

where x is the tortoise coordinate and $\mathcal{I}$ is given by initial data on the perturbation [Nollert 1999 Class. Quantum Grav. 16 R159].

Crucially, we notice that QNMs are given by zero of the same wronskian which gives Baxter's Q: *QNMs are Bethe roots!* [**arXiv:2112.11434**]

$$W(\omega_n) = W[\Psi_+, \Psi_-] = Q(\omega_n) = 0\,. \tag{9}$$

We thus prove that QNMs $\omega_n \propto \Lambda_n$ are given by quantization conditions on the Q, Y functions (and sometimes also T functions) and thanks to our gauge-integrability identifications

$$Q(\theta,P) = \exp\{2\pi i a_D(\hbar,u,\Lambda)\} = i\frac{\sinh A_D(\hbar,a,\Lambda)}{\sinh 2\pi i a}\,, \tag{10}$$

also on the gauge periods

$$A_D(\hbar,a,\Lambda_n) = i\pi n \tag{11}$$

$$2\pi i a_D(i\hbar,-u,\Lambda_n) = \pi i\left(n+\frac{1}{2}\right) \tag{12}$$

In this way we also give a mathematical explanation of the recently found connection between quasinormal modes and $\mathcal{N}=2$ supersymmetric gauge theories.

Moreover, it follows a new simple and effective method *to numerically compute the quasinormal modes: the Thermodynamic Bethe Ansatz (TBA)* for $\varepsilon = -2\ln Q$

$$\varepsilon(\theta) = \frac{16\sqrt{\pi^3}}{\Gamma(\frac{1}{4})^2}e^{\theta} - 2\int_{-\infty}^{\infty}\frac{\ln\left[1+\exp\{-\varepsilon(\theta')\}\right]}{\cosh(\theta-\theta')}\frac{d\theta'}{2\pi}\,. \tag{13}$$

which we compare with other standard methods and sometimes find convenient.

n	l	TBA	Leaver	WKB
0	0	$1.36912 - 0.504048i$	$1.36972 - 0.504311i$	$1.41421 - 0.5i$
0	1	$2.09118 - 0.501788i$	$2.09176 - 0.501811i$	$2.12132 - 0.5i$
0	2	$2.8057 - 0.501009i$	$2.80629 - 0.501000i$	$2.82843 - 0.5i$
0	3	$3.51723 - 0.500649i$	$3.51783 - 0.500634i$	$3.53553 - 0.5i$
0	4	$4.22728 - 0.500453i$	$4.22790 - 0.500438i$	$4.24264 - 0.5i$

The spacetimes for which we show this in all details are in the simplest $N_f=0$ case the D3 brane in the $N_f=1,2$ case a generalization of extremal Reissner-Nordström (extremally charged) BHs [**arXiv:2112.11434**]. Anyway, the ODE/IM construction can be realized still for the $SU(2)$ $N_f=3,4$ theories, which correspond to non-extremal and asymptotically flat or AdS BHs.

A lot of extension work, both technical and conceptual, still can be done!

Possible future developments

- Besides models extension, it is very intriguing to study the application of the integrability structure beyond the determination of the QNMs for the study of gravitational solutions (as an example, the *greybody factor* or *absorption coefficient* seems a ratio of Qs.).
- Very interestingly, AdS BHs ($SU(2)$ $N_f=4$ gauge theory) the 3-fold correspondence Integrability/Gauge/Gravity could becomes 4-fold through holographic duality.