

Integrability and cycles of deformed $\mathcal{N} = 2$ gauge theory

Based on arXiv:1908.08030 with Davide Fioravanti

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Introduction to deformed Seiberg-Witten theory

- The partition function for 4D $\mathcal{N} = 2$ SYM theories has been obtained through equivariant localisation techniques, [deforming spacetime](#) through two super-gravity parameters, ϵ_1 and ϵ_2 (the *Omega background*, needed for computing [instanton contributions](#)).
[Nekrasov:2004, Nekrasov-Okounkov:2006, Nekrasov:2009]
- When both $\epsilon_1, \epsilon_2 \rightarrow 0$, the logarithm of the partition function reproduces the [Seiberg-Witten prepotential](#) $\mathcal{F}_{\text{SW}}$. [Seiberg-Witten:1994]
- An intermediate limit which we will study is the [Nekrasov-Shatashvili \(NS\)](#): $\epsilon_1 = \hbar$, $\epsilon_2 \rightarrow 0$ [Nekrasov-Shatashvili:2009]. More specifically, having in mind the AGT corresponding Liouville field theory (and precisely its level 2 degenerate field equation), we may think of it as a quantisation/deformation of the quadratic SW differential for pure ($N_f = 0$) $SU(2)$ SYM which takes up the form of the [Mathieu equation](#)

$$-\frac{\hbar^2}{2} \frac{d^2}{dz^2} \psi(z) + [\Lambda^2 \cos z - u] \psi(z) = 0. \quad (1)$$

where u parametrizes the moduli space of vacua and Λ is a scaling parameter.

[Alday-Gaiotto-Tachikawa:2010; Gaiotto:2013; Awata-Yamada:2010]

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- The **deformed prepotential** $\mathcal{F}_{\text{NS}}$ (logarithm of the partition function) may be derived by eliminating u between the **two deformed cycles (periods)**

$$a(\hbar, u, \Lambda) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{P}(z; \hbar, u, \Lambda) dz, \quad a_D(\hbar, u, \Lambda) = \frac{1}{2\pi} \int_{-\arccos(u/\Lambda^2)-i0}^{\arccos(u/\Lambda^2)-i0} \mathcal{P}(z; \hbar, u, \Lambda) dz \quad (2)$$

of the quantum SW differential $\mathcal{P}(z) = -i \frac{d}{dz} \ln \psi(z)$. (In gauge theory also $a = 2\langle \tilde{\Phi} \rangle$, where $\tilde{\Phi}$ is the scalar field).

- In particular, we may expand asymptotically, around $\hbar = 0$, $\mathcal{P}(z) \doteq \sum_{n=-1}^{\infty} \hbar^n \mathcal{P}_n(z)$, and then the NS-deformed periods (modes) are **[Mironov-Morozov:2010]**

$$a^{(n)}(u, \Lambda) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{P}_{2n-1}(z; u, \Lambda) dz, \quad a_D^{(n)}(u, \Lambda) = \frac{1}{2\pi} \int_{-\arccos(u/\Lambda^2)-i0}^{\arccos(u/\Lambda^2)-i0} \mathcal{P}_{2n-1}(z; u, \Lambda) dz \quad (3)$$

- Alternatively, we can use Matone's formula connecting $\mathcal{F}_{\text{NS}}$, a , and u (still valid upon deformation). **[Matone:1995; Flume-Fucito-Morales-Poghossian:2004]**

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Introduction to ODE/IM with 2 irregular singularities

- Since early '80s , **2D Liouville field theory** has been recognised as the effective theory of **2D quantum gravity**. [**Polyakov:1981**] It is also an **integrable model**.
- In the approach of **ODE/IM correspondence** [**Dorey-Tateo:1999**; **Lukyanov-Bazhanov-Zamolodchikov:1999**; **Gaiotto-Moore-Neitzke:2010**], it was discovered by **Alexei Zamolodchikov** [**Zamolodchikov:2012**] that the solution $\psi(y)$ of the following ODE (**Generalized Mathieu Equation**)

$$\left\{ -\frac{d^2}{dy^2} + e^{2\theta}(e^{y/b} + e^{-yb}) + P^2 \right\} \psi(y) = 0, \quad (4)$$

can be used to construct Q , Y , and T functions and functional relations of the Liouville integrable model at L. coupling b and L. momentum P [**Zamolodchikov:2006**].

- w.r.t. "usual" ODE/IM, with polynomial potential, this equation has **2 irregular singularities** ($y \rightarrow \pm\infty$). Thus its study is interesting also for ODE/IM itself.

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- In ODE/IM, one defines the Q function

$$Q(\theta, P^2) = W[U_0, V_0] = -i \lim_{y \rightarrow +\infty} \frac{V_0(y; \theta)}{U_1(y; \theta)} . \quad (5)$$

where U_0 and V_0 are the solution of the GME (4) with b.c.

$$U_0(y) \simeq \frac{1}{\sqrt{2}} \exp\left\{-\theta/2 - y/4b\right\} \exp\left\{-2be^{\theta+y/2b}\right\} \quad \text{Re } y \rightarrow +\infty ; \quad (6)$$

$$V_0(y) \simeq \frac{1}{\sqrt{2}} \exp\left\{-\theta/2 + yb/4\right\} \exp\left\{-\frac{2}{b}e^{\theta-yb/2}\right\} \quad \text{Re } y \rightarrow -\infty . \quad (7)$$

- All Baxter's functions and functional relations of ODE/IM can be derived by considerations on linear relations among the solutions generated by the following **discrete symmetries** of the GME (4) (where $q = b + 1/b$)

$$\Lambda_b : \theta \rightarrow \theta + i\pi b/q \quad y \rightarrow y + 2\pi i/q , \quad \Omega_b : \theta \rightarrow \theta + i\pi/(bq) \quad y \rightarrow y - 2\pi i/q , \quad (8)$$

as $U_k = \Lambda_b^k U_0$ and $V_k = \Omega_b^k V_0$, with U_k invariant under Ω_b and V_k under Λ_b .

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- Def. $Y(\theta) = Q(\theta + i\pi a/2)Q(\theta - i\pi a/2)$, ($a = 1 - 2\frac{b}{q}$), **Y-system**

$$Y(\theta + i\pi/2)Y(\theta - i\pi/2) = \left(1 + Y(\theta + ia\pi/2)\right)\left(1 + Y(\theta - ia\pi/2)\right). \quad (9)$$

This functional equation can be inverted into the TBA for $\varepsilon(\theta) = -\ln Y(\theta)$

$$\varepsilon(\theta) = \frac{8\sqrt{\pi^3}q}{\Gamma(\frac{b}{2q})\Gamma(\frac{1}{2bq})}e^\theta - \quad (10)$$

$$- \int_{-\infty}^{\infty} \left[\frac{1}{\cosh(\theta - \theta' + ia\pi/2)} + \frac{1}{\cosh(\theta - \theta' - ia\pi/2)} \right] \ln [1 + \exp\{-\varepsilon(\theta')\}] \frac{d\theta'}{2\pi}, \quad (11)$$

with boundary condition $\varepsilon(\theta, P^2) \simeq +4qP\theta$, $P > 0$, at $\theta \rightarrow -\infty$.

- Def. two (dual under $b \rightarrow 1/b$) T functions through **TQ-relations** ($p = \frac{b}{q}$)

$$T(\theta)Q(\theta) = Q(\theta + i\pi p) + Q(\theta - i\pi p) \quad \tilde{T}(\theta)Q(\theta) = Q(\theta + i\pi(1-p)) + Q(\theta - i\pi(1-p)), \quad (12)$$

- It can be derived in ODE/IM also the **periodicity of T**

$$T(\theta + i\pi(1-p)) = T(\theta) \quad \tilde{T}(\theta + i\pi p) = \tilde{T}(\theta). \quad (13)$$

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Integrability-Gauge fundamental correspondence

- The **self-dual** ($b = 1$) GME is known in literature as **modified Mathieu equation**:

$$\left\{ -\frac{d^2}{dy^2} + 2e^{2\theta} \cosh y + P^2 \right\} \psi(y) = 0 , \quad (14)$$

- This equation can be related to the Mathieu equation which quantises the SW differential in the NS limit (1) by the independent variable change $z = -iy - \pi$ and the **parameters correspondence**

$$\frac{\hbar}{\Lambda} = \frac{\epsilon_1}{\Lambda} = e^{-\theta} \quad \frac{u}{\Lambda^2} = \frac{1}{2} \frac{P^2}{e^{2\theta}} . \quad (15)$$

- We (**D. Fioravanti and D. Gregori - arXiv:1908.08030**) have found that the two deformed Seiberg-Witten cycle periods for **pure** ($N_f = 0$) **SU(2) $\mathcal{N} = 2$ supersymmetric gauge theory** are connected to the Baxter's Q and T functions of the **Liouville integrable model at the self dual point** by the very simple relations:

$$Q(\theta, P^2) = \exp 2\pi i a_D(\hbar, u) , \quad (16)$$

$$T(\theta, P^2) = 2 \cos 2\pi a(\hbar, u) . \quad (17)$$

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Proof of the fundamental identification $Q = \exp 2\pi i a_D$

We have proven relation (16) analytically by studying the properties of the solution $\mathcal{P}(y) = -i \frac{d}{dy} \ln \psi(y)$ of the Riccati equation

$$\mathcal{P}^2(y, \hbar, u) - i \frac{d\mathcal{P}(y, \hbar, u)}{dy} = -\left(\frac{2u}{\hbar^2} + \frac{2\Lambda^2}{\hbar^2} \cosh y\right), \quad (18)$$

with boundary condition given by the (double) limit $y \rightarrow +\infty$ of the Seiberg-Witten leading $\hbar \rightarrow 0$ order: $\mathcal{P}_{-1}(y) = -i \sqrt{\frac{2u}{\hbar^2} + \frac{2\Lambda^2}{\hbar^2} \cosh y} \sim -i \frac{\Lambda}{\hbar} e^y$, as $y \rightarrow +\infty$.

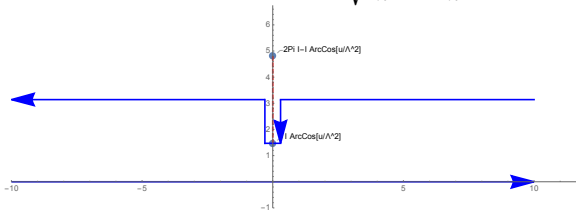


Figure: Integration contour in the y complex plane for the proof of relation (16).

$$\underbrace{\int_{-\arccos(u/\Lambda^2)-i0}^{+\arccos(u/\Lambda^2)-i0} \mathcal{P}(z) dz}_{2\pi i a_D(\hbar, u)} = \underbrace{\int_{-\infty}^{+\infty} \mathcal{P}_{reg}(y) dy}_{\ln Q(\theta, P^2)}, \quad (19)$$

$$\mathcal{P}_{reg}(y) = \mathcal{P}(y) + 2ie^\theta \cosh \frac{y}{2} - \frac{i}{4} \tanh y. \quad (20)$$

Proof of the fundamental identification $T = 2 \cos 2\pi a$

- [Alexei Zamolodchikov](#) proved a relation between the $b = 1$ T and the [Floquet index](#) ν of the Mathieu equation, such that $\psi(z + 2\pi) = e^{2\pi\nu}\psi(z)$. [[Zamolodchikov:2012](#)]

$$T(\theta, P^2) = 2 \cosh 2\pi\nu = 2 [1 - 2\Delta(0) \sin^2 \pi P] . \quad (21)$$

- ν was computed from the Hill determinant $\Delta(0)$ and T was computed through the analytic continuation of the TBA (11) for $\ln Q = -\frac{1}{2}\varepsilon$, by the TQ relation (12).

$$\Delta(0) = \lim_{n \rightarrow \infty} \det \mathcal{A}_n(0) \quad \mathcal{A}_n(\nu) = \begin{pmatrix} 1 & \xi_{-n} & 0 & & & \\ \xi_{-n+1} & 1 & \xi_{-n+1} & & & \\ & & \vdots & \ddots & & \\ & & & & 0 & \xi_0 & 1 & \xi_0 \\ & & & & & & \ddots & \vdots \\ & & & & & & & \xi_n & 1 \end{pmatrix} . \quad (22)$$

$$\xi_n = \frac{e^{2\theta}}{P^2 - n^2}$$

- He did not recognise that $\nu = ia$ (NS limit of $\mathcal{N} = 2$ SYM not yet existing), but we did it, thus establishing another connection to [gauge theory](#).

Proof of the fundamental identification $T = 2 \cos 2\pi a$

- Alexei Zamolodchikov proved a relation between the $b = 1$ T and the Floquet index ν of the Mathieu equation, such that $\psi(z + 2\pi) = e^{2\pi\nu}\psi(z)$. [Zamolodchikov:2012]

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Proof of the fundamental identification $T = 2 \cos 2\pi a$

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Gauge TBA

- The self-dual Liouville **QQ relation** for $Q(\hbar, u) \equiv Q(\theta, u) \equiv Q(\theta, P^2)$ translates in the gauge variables as

$$1 + Q^2(\theta, u) = Q(\theta - i\pi/2, -u)Q(\theta + i\pi/2, -u) \quad (23)$$

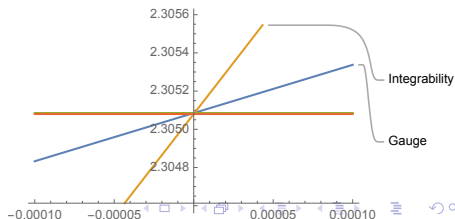
and can be inverted to obtain a **TBA for the dual period** ($e^{-\varepsilon(\theta, u)} = Q^2(\theta, u)$)

$$\varepsilon(\theta, u) = -4\pi i a_D^{(0)}(u) \frac{e^\theta}{\Lambda} - 2 \int_{-\infty}^{\infty} \frac{\ln [1 + \exp\{-\varepsilon(\theta', -u)\}]}{\cosh(\theta - \theta')} \frac{d\theta'}{2\pi} \quad (24)$$

- The TBA enjoys the $\mathbb{Z}_2$ **R-symmetry** of the moduli space $u \leftrightarrow -u$, which is nothing but the **discrete symmetry** Λ_1 or Ω_1 used for the **ODE/IM construction**.
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$$\varepsilon(\theta_0, u) = \varepsilon(\theta_0, P^2) \text{ when } P^2 = 2ue^{2\theta_0}/\Lambda^2, \quad (25)$$

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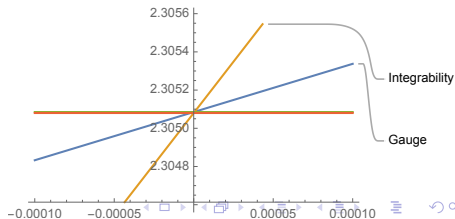
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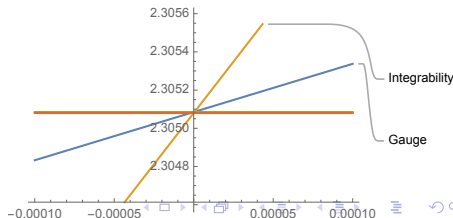
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- The self-dual Liouville TQ relation and T periodicity relation read in gauge variables

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- These relations appear to be a non-perturbative exact generalization of the perturbative $\mathbb{Z}_2$ R -symmetry relations for the periods [Bilal-Ferrari:1996, Basar-Dunne:2015]

$$a_D^{(n)}(-u) = i(-1)^n \left[-\operatorname{sgn}(\operatorname{Im} u) a_D^{(n)}(u) + a^{(n)}(u) \right], \quad (28)$$

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Perturbative periods and local integrals of motion

- The $\theta \rightarrow +\infty$ asymptotic expansions of Q in finite gauge u and integrability P variables are

$$Q(\theta, u) \doteq \exp \left\{ 2\pi i \sum_{n=0}^{\infty} e^{\theta(1-2n)} \Lambda^{2n-1} a_D^{(n)}(u, \Lambda) \right\} \quad , \quad (30)$$

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where $l_{2n-1}(b=1, P^2) = \sum_{k=0}^n \Upsilon_{n,k} P^{2k}$

- Since in Seiberg Witten theory u is finite as $\theta \rightarrow +\infty$, it is necessary that also $P^2(\theta) = 2 \frac{u}{\Lambda^2} e^{2\theta} \rightarrow +\infty$. In this double limit, the LIMs resum to the perturbative periods.

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- We have derived from the series (32) the **quantum Picard-Fuchs equations at all perturbative orders** (through an algorithm) for $a^{(n)}$ and $a_D^{(n)}$. For instance

$$\left\{ (u^2 - \Lambda^4) \frac{\partial^2}{\partial u^2} + 4u \frac{\partial}{\partial u} + \frac{5}{4} \right\} a_D^{(1)}(u, \Lambda) = 0, \quad (33)$$

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- In conclusion, the powerful ODE/IM correspondence has been revealing a **very suggestive connexion** between the **quantum integrable models** and ϵ_1 -deformed Seiberg-Witten $\mathcal{N} = 2$ supersymmetric gauge theories.
- The ODE/IM correspondence yields a **natural quantisation scheme** for general SW theory: (the suitable power of) the SW differential becomes quantised as differential operator or oper whose cycles (periods) or monodromies are encoded into the connexion coefficients (for instance, of the ODE/IM).
- It would be also interesting to explore:
 - the implications for the cycles and periods as described in [Bourgine-Fioravanti:2018A,B];
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THANK YOU!