New developments in N=2 supersymmetric gauge theories: integrability, black holes

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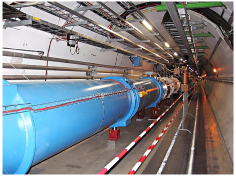
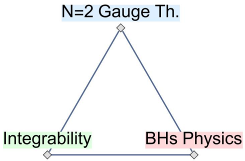
with Davide Fioravanti and Hongfei Shu

On extended supersymmetric gauge theories

- $\mathcal{N} = 1$ SUSY gauge theory:
  - conjectured supersymmetric partner: **pairs** of particles;
  - not yet found at LHC, but **not excluded**.
- Extended $\mathcal{N} = 2$, $\mathcal{N} = 4$ SUSY gauge theories:
  - quartets and octets of particles, mathematically richer;
  - although more distant from experiments, can give **insights on conceptual mechanism of QFT** (e.g. confinement);
  - **integrability** has longly **survived also without SUSY**;
  - also **many applications** beyond particle physics:
    - $\mathcal{N} = 4$ CFT holographic dual to String Theory on AdS spacetime;
    - $\mathcal{N} = 2$ also allows non CFTs, recently new insight on it:





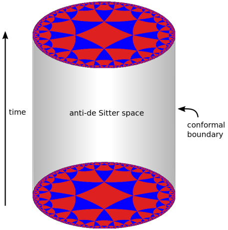


Figure: (Wikicommons/J.Herzog,A.Dunkel)

Outline

1. Introduction I ($\mathcal{N} = 2$ gauge theory)
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   - The ODE/IM correspondence
   - A new gauge-integrability correspondence
3. Introduction II (black holes perturbation theory)
   - On quasinormal modes
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4. Integrability for black holes
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   - A new method of computation of QNMs (I)
   - $\mathcal{N} = 2$ gauge theory and black holes
   - A new method of computation of QNMs (II)
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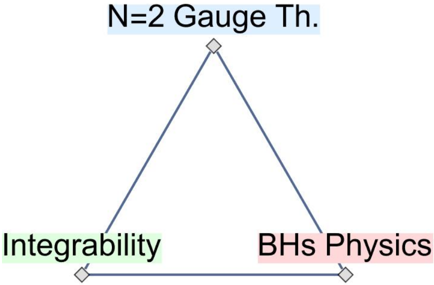
5. Conclusions and perspectives



Summary of $\mathcal{N} = 2$ gauge (Seiberg-Witten) theory

- Seiberg-Witten (SW) theory is the **effective theory** for $\mathcal{N} = 2$ gauge theory.
- Its lagrangian is expressed in terms of a SW prepotential $\mathcal{F}_{SW}$

$$\mathcal{L} = \frac{1}{16\pi}\mathrm{Im}\int d^4x \left[\frac{1}{2}\int d^2\theta\, \mathcal{F}''_{SW}(\phi) W^\alpha W_\alpha + \int d^2\theta d^2\bar{\theta}\phi^\dagger \mathcal{F}'_{SW}(\phi)\right] . \tag{1}$$

- $\mathcal{F}_{SW}$ can be obtained from **classical SW periods** $a^{(0)}$, $a_D^{(0)}$

$$a_D^{(0)} = \frac{\partial \mathcal{F}}{\partial a^{(0)}} , \tag{2}$$

$$a^{(0)}(u, \mathbf{m}, \Lambda) = \oint_{\mathcal{A}} \lambda(x, u, \mathbf{m}, \Lambda)\, d\, x , \qquad a_D^{(0)}(u, \mathbf{m}, \Lambda) = \oint_{\mathcal{B}} \lambda(x, u, \mathbf{m}, \Lambda)\, d\, x , \tag{3}$$

which are integrals of a SW differential λ related to the SW hyperelliptic curve

$$\lambda \sim \sqrt{x^3 + c_2 x^2 + c_1 x + c_0} . \tag{4}$$

Quantum Seiberg-Witten curve and gauge periods

- To compute **instanton contributions** spacetime is deformed by two complex parameters ϵ_1, ϵ_2 into the Ω-**background**.
- More easily connected to integrability and gravity is the **Nekrasov-Shatashvili (NS) limit** $\epsilon_2 \to 0, \epsilon_1 = \hbar \neq 0$, in which the **quantum SW curves** are just ODEs in $y \sim -\ln x$

$$-\hbar^2 \frac{d^2}{dy^2}\psi(y) + [U(y; \mathbf{m}, \Lambda) - u]\,\psi(y) = 0\,. \tag{5}$$

- The NS prepotential $\mathcal{F}_{NS}$ has as leading $\hbar \to 0$ order the SW prepotential

$$\mathcal{F}_{NS} = \frac{1}{\hbar}\mathcal{F}_{SW} + O(\hbar) + O(e^{-\frac{1}{\hbar}}) \qquad \hbar \to 0 \tag{6}$$

and is obtained define **quantum exact periods** a, a_D which are either
  - **exact integrals** of $\mathcal{P}(y) = -i\frac{d}{dy}\ln\psi(y)$
  - series in Λ with coefficients given by **combinatorics on Young diagrams** (of gauge repr.)

Integrability, generally speaking

- Very broadly speaking, integrability can be considered as the study of **non-linear** phenomena in nature in a quantitative **exact way (non perturbative)**.
- Integrable structures appear also in **4D SUSY gauge theories**, typically with $\mathcal{N} = 4$ supersymmetry but also more recently with $\mathcal{N} = 2$.
- The hallmark of quantum integrability is the presence of **infinite (local) integrals of motion commuting** with each other

$$[\mathbf{I}_{2n-1}, \mathbf{I}_{2m-1}] = 0\,, \tag{7}$$

which are also asymptotic expansion coefficients of the **Baxter's Q operator**

$$\ln \mathbf{Q}(\theta) \simeq -C_0 e^{\theta} - \sum_{n=1}^{\infty} e^{\theta(1-2n)} C_n \mathbf{I}_{2n-1} \qquad \theta \to +\infty\,. \tag{8}$$

ODE/IM correspondence approach to integrability

- ODE/IM correspondence is an approach to integrability [arXiv:9812211, 9812247, 9906219], in which the Q **function** (vacuum eigenvalue of Q operator) is typically the wronskian of the regular solutions at different singular points

$$Q = W[\psi_+, \psi_-] \qquad \text{with} \qquad \psi_\pm(y) \to 0 \quad y \to \pm\infty \tag{9}$$

of some ODE, like (for self-dual Liouville IM):

$$-\frac{d^2}{dy^2}\psi + \left[e^{2\theta}(e^y + e^{-y}) + P^2\right]\psi = 0\,. \tag{10}$$

- One can derive also T, Y **functions** as well as the **functional and integral equations** they satisfy.
- ODE/IM is an **elegant approach** to integrability which **allows to apply it to very different physical theories!**

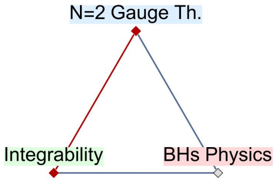
Basic gauge-integrability identifications

- Using ODE/IM correspondence, we **connected** the **basic integrability functions** to the gauge exact quantum periods a, a_D, proving relations like (for self-dual Liouville and $SU(2)$ $N_f=0$):

$$Q(\theta,P)=\exp\frac{2\pi i}{\hbar}a_D(\hbar,u,\Lambda_0)$$
$$T(\theta,P)=2\cos\left[\frac{2\pi}{\hbar}a(\hbar,u,\Lambda_0)\right]\,, \tag{11}$$

under the parameters correspondence

$$\frac{\hbar}{\Lambda_0}=\frac{\epsilon_1}{\Lambda_0}=e^{-\theta}\qquad\frac{u}{\Lambda_0^2}=\frac{1}{2}P^2e^{-2\theta}\,. \tag{12}$$



- These for the self-dual Liouville IM and $SU(2)$ $N_f=0$ gauge theory [**arXiv:1908.08030**] but also similar ones for the Perturbed Hairpin IM and $SU(2)$ $N_f=1,2$ [**arXiv:2208.14031**] and $SU(3)$ $N_f=0$ gauge theories [**arXiv:1909.11100**], work in progress in $SU(2)$ $N_f=3,4$.

New results for both gauge theory and integrability

- This basic identification allowed us to find several **new results for both sides**, for instance:
  1. an interpretation of the integrability functional relations as **new exact R-symmetry relations for the periods**;
  2. an exact non linear integral equation (**Thermodynamic Bethe Ansatz, TBA**) for the **gauge (dual) periods**;
  3. **new formulas for the local integrals of motion** in terms of gauge periods
  4. **exact computation of the $\mathcal{N}=2$ gauge prepotential $\mathcal{F}_{NS}$** (effective low energy theory)
- For instance, the **Baxter's TQ relation**

$$T(\theta,u)=\frac{Q(\theta-i\pi/2,-u)+Q(\theta+i\pi/2,-u)}{Q(\theta)} \tag{13}$$

turns out to be a new exact R-symmetry $u\leftrightarrow -u$ relation for the periods, reducing to the known asymptotic ones in the limit $\hbar\to 0 \iff \theta\to\infty$ (for the gauge periods expansion modes $a^{(n)}$, $a_D^{(n)}$)

$$a_D^{(n)}(-u)=i(-1)^n\left[-\mathrm{sgn}(\mathrm{Im}\,u)a_D^{(n)}(u)+a^{(n)}(u)\right]\,. \tag{14}$$

Three different physical theories, same mathematics!

$$-\frac{\hbar^2}{2}\frac{d^2}{dy^2}\psi(y)+[\Lambda^2\cosh y+u]\psi(y)=0 \quad \text{N=2 GAUGE TH. } (SU(2)\ N_f=0) \tag{15}$$

$$\frac{\hbar}{\Lambda_0}=\frac{\epsilon_1}{\Lambda_0}=e^{-\theta} \qquad \frac{u}{\Lambda_0^2}=\frac{1}{2}P^2e^{-2\theta} \qquad \Downarrow \tag{16}$$

$$-\frac{d^2}{dy^2}\psi(y)+\left[e^{2\theta}(e^y+e^{-y})+P^2\right]\psi(y)=0 \quad \text{INTEGRABILITY (sd-Liouville)} \tag{17}$$

$$r=Le^{y/2} \quad \omega L=-2ie^{\theta} \quad P=\frac{1}{2}(l+2) \qquad \Downarrow \qquad \phi(r)=e^{y/2}\psi(y) \tag{18}$$

$$\frac{d^2\phi}{dr^2}+\left[\omega^2\left(1+\frac{L^4}{r^4}\right)-\frac{(l+2)^2-\frac{1}{4}}{r^2}\right]\phi(r)=0 \quad \text{BLACK HOLES PERT. (D3 brane)} \tag{19}$$

Mathematical definition of quasinormal modes

- **Perturbations of the black holes (BHs)** metric or fields turns out to be **solutions Φ of some PDEs** of the form

$$\left\{+\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial y^2}+U(y)\right\}\Phi(t,y)=0\,, \tag{20}$$

- By standard techniques (Laplace tr. $\rightarrow$ non-hom. ODE $\rightarrow$hom. ODE) we can express the perturbation Φ as an expansion over the **quasinormal modes (QNMs)** frequencies ω_n

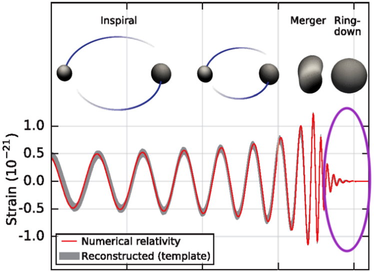
$$\Phi(t,y)=\sum_n e^{i\omega_n t}\text{Res}\left(\frac{1}{W(s)}\right)\Bigg|_{\omega_n}\int_{-\infty}^{\infty}\psi_-(\omega_n,y_<)\psi_+(\omega_n,y_>)\mathcal{I}(\omega_n,y')\,dy'\,. \tag{21}$$

- QNMs ω_n are so found to be **zeros of wronskian of the fundamental regular solutions** at $y\rightarrow\pm\infty$ of the ODE:

$$W[\psi_+,\psi_-](\omega_n)=0\,,\qquad \text{with}\qquad \psi_\pm(y)\rightarrow 0\quad y\rightarrow\pm\infty\,. \tag{22}$$

Colliding BHs and quasinormal modes

- A **black hole collision** can be divided in 3 phases: **inspiral, merger** and **ringdown**.

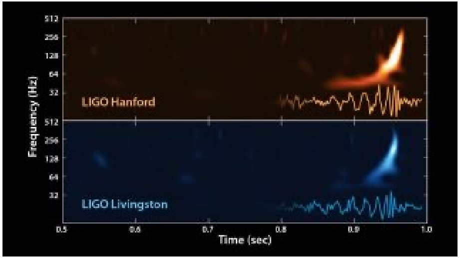


- The **quasinormal modes** are responsible for the **damped oscillations** appearing, for example, in the **ringdown phase** of two colliding BH and have a direct connection to **gravitational waves observations**.

The new era of gravitational phenomenology

- In the last few years, **gravitational waves** detections and **black hole imaging** have opened the doors of **gravitational phenomenology**. [Mayerson:2020]

- Finally, we can fully scientifically investigate whether real astrophysical black holes show **deviations from general relativity (GR)**, such as **horizon scale structure**.



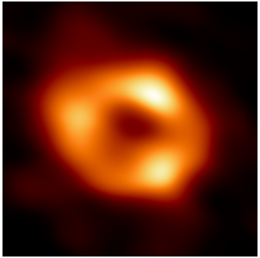


Figure: LIGO, Event Horizon

- Since BHs are the simplest and most perfect objects in gravity, someone asks:
  Could BHs hold the same surprises that the electron and the hydrogen atom did when they started to be experimentally probed? [Cardoso,Pani:2017]

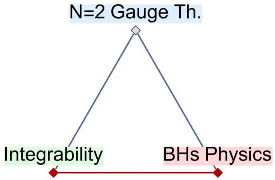
Baxter's Q function and quasinormal modes

- We proved that the **Baxter's Q function** is defined **precisely as the wronskian of the regular solutions**

$$Q(\theta) = W[\psi_+, \psi_-], \quad \text{with} \quad \psi_\pm \to 0 \quad y \to \pm\infty \tag{23}$$

- Crucially, the **QNMs condition** (22) translates into

$$Q(\theta_n) = 0\,, \tag{24}$$



namely the **zeros of the Baxter's Q function** which are called **Bethe roots**.

- By defining the Y **function** as some quadratic function of Q we can prove an equivalent condition

$$Y(\theta) = Q^2(\theta) \qquad Y(\theta_n - i\pi/2) = -1 \tag{25}$$

Thermodynamic Bethe Ansatz

- The Y functions satisfies a functional equation (Y system) which can be inverted into the **Thermodynamic Bethe Ansatz (TBA)** for $\varepsilon(\theta) = -\ln Y(\theta)$ (here for $SU(2)$ $N_f = 0$):

$$\varepsilon(\theta) = \frac{16\sqrt{\pi^3}}{\Gamma(\frac{1}{4})^2} e^{\theta} - 2 \int_{-\infty}^{\infty} \frac{\ln\left[1 + \exp\{-\varepsilon(\theta')\}\right]}{\cosh(\theta - \theta')} \frac{d\theta'}{2\pi} \,. \tag{26}$$

  with $\varepsilon(\theta, P) \simeq 8P\theta$, $P > 0$ as $\theta \to -\infty$.

- The **QNMs condition** in gravity variables **reads also as**

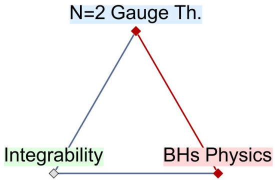
$$\varepsilon(\theta_{n'} - i\pi/2) = -i\pi(2n' + 1) \,. \tag{27}$$

- Through this TBA and quantization condition we have found a **new way to numerically compute QNMs**.

A surprising application

- In the last two years, a **surprising connection** between $\mathcal{N} = 2$ supersymmetric $SU(2)$ NS gauge theories and black holes (BHs) perturbation theory has emerged [**arXiv:2006.06111**, 2105.04245, 2105.04483, arXiv:2109.09804, ...].
- It was first found that **quantization conditions on the gauge periods** a, a_D allow to **compute the QNMs** ω_n spectrum of black holes from gauge theory methods.

$$\frac{1}{\hbar} a_D(\hbar, u, m, \Lambda) = \frac{i}{2}\left(n + \frac{1}{2}\right) , \qquad n = 0, 1, 2, \dots \tag{28}$$



- The importance of this result is manifold, as it:
  - constitutes a **novel analytic characterization of QNMs**;
  - has already allowed to **find new results** for the BHs theory;
  - constitutes an **unexpected application of Supersymmetry**.

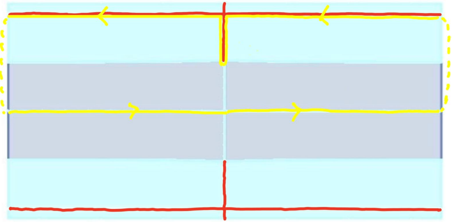
Quantum gauge B period from TBA

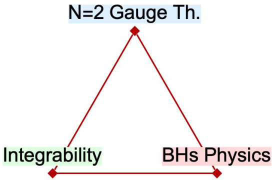
- We proved a **relation between the pseudoenergy** $\epsilon = -\ln Y = -2\ln Q$ **and the gauge periods**

$$Q = \exp \frac{2\pi i}{\hbar} a_D \iff \varepsilon = -\frac{4\pi i}{\hbar} a_D \,. \tag{29}$$

- We can prove it analytically by **Cauchy theorem relating the different integration contours** in the complex plane (in red the branch cuts).
- We see that from our formalism **it follows immediately that QNMs are given by quantization conditions on the gauge periods**

$$\frac{1}{\hbar} a_D(\hbar, u, \Lambda) = \frac{i}{2}\left(n' + \frac{1}{2}\right) \,. \tag{30}$$





Comparison of methods of computation of quasinormal modes

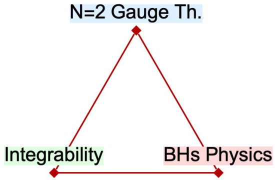
n	l	TBA	Leaver	WKB
0	1	$0.869623 - 0.372022i$	$0.868932 - 0.372859i$	$0.89642 - 0.36596i$
0	2	$1.477990 - 0.368144i$	$1.477888 - 0.368240i$	$1.4940 - 0.36596i$
0	3	$2.080200 - 0.367076i$	$2.080168 - 0.367097i$	$2.0916 - 0.36596i$
0	4	$2.680363 - 0.366637i$	$2.680350 - 0.366642i$	$2.6893 - 0.36596i$

Table: Comparison of QNMs in different methods ($n' = 0$, $SU(2)$ $N_f = 2$ gauge theory with $\Sigma_1 = \Sigma_3$).

- Computing QNMs has been typically quite **laborious**, also because of their **few exact analytic characterizations**.
- The standard analytic method is the one with the **continued fractions** by Leaver.
- Application of $\mathcal{N} = 2$ **gauge theory** is a new analytic characterization of QNMs, but it requires a **nontrivial re-summation procedure** for $\omega_n \sim \Lambda \sim 1$.
- Our **integrability** exact method is **direct and simple**, but now we have developed for just a **few models** (see below).

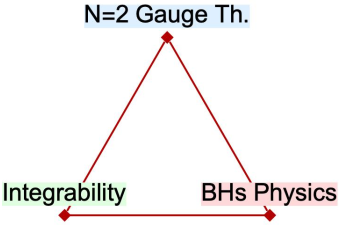
Other applications of integrability

- Through ODE/IM can also define also **Baxter's T, $\tilde{T}$ functions** as wronskians of solutions at $y \to \pm\infty$ and derive TQ and T periodicity functional relations. From this:
  - since we have proven a relation between T and a gauge period, we give also a proof of the **alternative quantization condition for QNMs on** a for the $SU(2)$ $N_f = 0$ theory:
  - moreover we find that the **Couch-Torrence symmetry** corresponds to T **self-duality**.
- We notice that **much of the BH theory seems to go in parallel to the ODE/IM correspondence** construction and its 2D integrable field theory interpretation, beyond the determination of QNMs!
  - For instance also the **greybody factor** that parametrizes the **Hawking radiation** seems to be **ratio of** Q**s**.



Conclusions

- We have shown how **2D integrable models** when studied in **ODE/IM correspondence** approach can find
  - a natural connection to (deformed) $\mathcal{N} = 2$ **supersymmetric gauge theory**
  - as well as to **black hole perturbation theory**
  - and shed light on the **relation** recently found **between the two**.
- This allows to
  - **find new results** an all three sides of the correspondence
  - at the **non-perturbative exact level**.
- In these new directions **much extension work in either breadth and depth** can still be done.



Present limitations and possible future developments

- What we have described holds for the **generalization of extremal charged BHs** (intersection of 4 stacks of D3 branes) and it corresponds to $SU(2)$ $N_f = 2, 1, 0$.
- However, from the **generality of construction** it is manifest that our method **should apply many other theories**. We can list
  1. $SU(2)$ $N_f = (2, 0)$ gauge theory, associated to D1D5 fuzzballs and CCLP 5D BHs, etc.;
  2. $SU(2)$ $N_f = 3$ gauge theory, , associated to asymptotically flat Kerr-Newman BH;
  3. $SU(2)$ $N_f = 4$ or $SU(2)$ quivers, associated to asymptotically AdS BHs.
- Besides **model generalization**, it is very intriguing to study the application of the integrability structure **beyond the determination of the QNMs** for the study of gravitational solutions (greybody factor, etc.).
- For **AdS BHs** the 3-fold correspondence Integrability/Gauge/Gravity could become 4-fold through **holographic duality to CFT**.

Thank you
for your attention!