Integrability tools and insights into black holes' ringdown

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with D. Fioravanti, R. Mahanta, M. Rossi, H. Shu

From quantum integrability to gravitational waves

On quantum integrability

- Integrability can be considered as the study of **non-linear** phenomena in nature in a quantitative **exact, non perturbative** way.
- The hallmark of quantum integrability is the presence of **infinite (local) integrals of motion commuting** with each other

$$[\mathbf{I}_n, \mathbf{I}_m] = 0\,. \qquad (1)$$

What objects are more general and abstract?

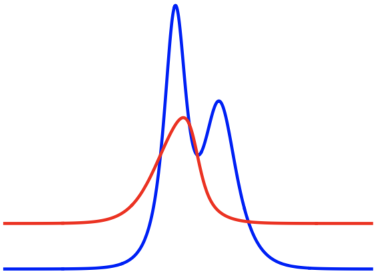


Figure: Solitons of mKdV (Wolfram Demo)

- The IM I_{2n-1} are also asymptotic expansion coefficients, for large spectral parameter $\lambda = e^{\theta}$, of the **Baxter's** Q, T **operators**

$$\ln \mathbf{Q}(\theta) \simeq -C_0 e^{\theta} - \sum_{n=1}^{\infty} e^{-n\theta} C_n \mathbf{I}_n \qquad \theta \to +\infty\,, \tag{2}$$

$$[\mathbf{Q}(\theta), \mathbf{T}(\theta)] = 0\,, \tag{3}$$

their eigenvalues being called Q, T functions.

- Q, T functions satisfy **functional relations**, like **quantum wronskian** QQ or **Baxter's** TQ (here for Sinh-Gordon IM)

$$Q(\theta - i\pi/2)Q(\theta + i\pi/2) - Q(\theta - i\pi a/2)Q(\theta + i\pi a/2) = 1\,, \tag{4}$$

$$T(\theta)Q(\theta) = Q(\theta + i\pi(a+1)/2) + Q(\theta - i\pi(a+1)/2)\,. \tag{5}$$

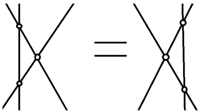
- Given suitable **asymptotic behaviours** these functional relations have **unique solutions**.

The ODE/IM correspondence

- The **ODE/IM correspondence** is an approach to integrability, in which the Q, T functions and all the (QQ, TQ...) **integrable structures** of some Integrable Model (IM) can be **derived from Ordinary Differential Equation** (ODE), even without knowing the corresponding IM.

$$\left[-\frac{d^2}{dx^2}+U\right]\psi = E\psi$$

ODE	IM
Spectral determinant	Q
Stokes multipliers	T_k
Central Connect. rels.	QQ relation
Lateral Connect. rels.	TQ, T-syst
ODE parameters	IM parameters
ODE ind. var. x	... cf. below



ODE/IM allows to apply integrability it to very different physical theories!

Three different physical theories, same mathematics!

$$-\frac{d^2}{dy^2}\psi(y) + \left[2e^{2\theta}\cosh y + P^2\right]\psi(y) = 0 \qquad \text{INTEGRABILITY (sd-Liouville)} \tag{6}$$

$$\frac{\hbar}{\Lambda_0} = \frac{\epsilon_1}{\Lambda_0} = e^{-\theta} \qquad \frac{u}{\Lambda_0^2} = \frac{1}{2}P^2 e^{-2\theta} \qquad \Downarrow$$

$$-\frac{\hbar^2}{2}\frac{d^2}{dy^2}\psi(y) + [\Lambda^2\cosh y + u]\psi(y) = 0 \qquad \text{N=2 GAUGE TH. } (SU(2)\ N_f = 0) \tag{7}$$

$$r = Le^{y/2} \quad \omega L = -2ie^{\theta} \quad P = \frac{1}{2}(l+2) \qquad \Downarrow \qquad \phi(r) = e^{y/2}\psi(y)$$

$$\frac{d^2\phi}{dr^2} + \left[\omega^2\left(1 + \frac{L^4}{r^4}\right) - \frac{(l+2)^2 - \frac{1}{4}}{r^2}\right]\phi(r) = 0 \qquad \text{BLACK HOLES PERT. (D3 brane)} \tag{8}$$

- We have shown this **triality** generally for Mathieu , Doubly Confluent Heun , Confluent Heun ODEs .

Black holes perturbation theory

- For perturbations of the **Schwarschild BH** metric $g^{(0)}_{\mu\nu}$, **Einstein equations**, after **linearization** and **gauge fixing** reduce to just a couple of much simpler PDEs:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{9}$$

$$\Downarrow \quad h_{\mu\nu} = g_{\mu\nu} - g^{(0)}_{\mu\nu} \Rightarrow (Z_l, Q_l)$$

1) for the **axial** perturbations (phase $(-1)^{l+1}$), the **Regge-Wheeler** equation

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \left(1 - \frac{2M}{r}\right) \left(\frac{l(l+1)}{r^2} - \frac{6M}{r^3} \right) \right] Q_l(t,x) = 0\,; \tag{10}$$

2) for the **polar** perturbations (phase $(-1)^l$), the **Zerilli** equation

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \left(1 - \frac{2M}{r}\right) \left(\frac{72M^3}{r^5 \lambda(r)^2} + (\dots) \frac{12M}{r^3 \lambda(r)^2} \left(1 - \frac{3M}{r}\right) + (\dots) \frac{1}{r^2 \lambda(r)} \right) \right] Z_l(t,x) = 0, \tag{11}$$

where $\lambda(r) = l(l+1) - 2 + 6M/r$ and $r = r(x)$, with x tortoise coordinate.

From PDEs to ODEs

- The **Laplace transform** $\hat{\Psi}$ of $\Phi = Z_l \vee Q_l$ satisfies the non-homogeneous ODE

$$\hat{\Psi}(s,x) = \int_0^\infty e^{-st}\Phi(t,x)\,dt\,, \qquad \left\{-\frac{\partial^2}{\partial x^2} + U(x) + s^2\right\}\hat{\Psi}(s,x) = -\mathcal{I}(s,x)\,, \tag{12}$$

where $\mathcal{I}(s,x) = -s\Psi(t,x)|_{t=0} - \partial_t\Psi(t,x)|_{t=0}$. The corresponding **homogeneous** equation is the **ODE we will study**:

$$\left\{-\frac{\partial^2}{\partial x^2} + U(x) + s^2\right\}\Psi(s,x) = 0\,. \tag{13}$$

- We consider **solutions** of (13) **regular** for $x \to \pm\infty$, that is infinity and horizon (with $\mathrm{Re}\, s > 0$)

$$\Psi_+(s,x) \sim e^{-sx}\,, \qquad x \to +\infty\,, \qquad \Psi_-(s,x) \sim e^{sx}\,, \qquad x \to -\infty\,. \tag{14}$$

- The Laplace tr. $\hat{\Psi}$ is given by the **Green function** G as (with $x_< = \min(x',x)$, $x_< = \max(x',x)$)

$$\hat{\Psi}(s,x) = \int_{-\infty}^\infty G(s,x,x')\mathcal{I}(s,x')dx'\,, \qquad G(s,x,x') = \frac{1}{W[\Psi_-,\Psi_+]}\Psi_-(s,x_<)\Psi_+(s,x_>)\,. \tag{15}$$

Quasinormal modes

- Finally taking the anti-Laplace transform of $\hat{\Psi}$ and setting $s = i\omega$ we get the **original perturbation** $\Phi = Z_l \vee Q_l$ as

$$\Phi(t,x) = \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} e^{st}\hat{\Psi}(s,x)\, ds \tag{16}$$

$$= \sum_n e^{i\omega_n t}\mathrm{Res}\left(\frac{1}{W(s)}\right)\Bigg|_{\omega_n}\int_{-\infty}^{\infty}\Psi_-(\omega_n, x_<)\Psi_+(\omega_n, x_>)\mathcal{I}(\omega_n, x')\, dx' + (\text{other contributions})\,. \tag{17}$$

- The **quasinormal modes (QNMs)** are the complex frequencies (real frequencies and damping times) ω_n, which corresponds also to **zeros of the ODE Wronskian**

$$W(\omega_n) \equiv W[\Psi_-, \Psi_+](\omega_n) = 0\,, \tag{18}$$

with $W[\Psi_-, \Psi_+] = \Psi_-(x)\Psi_+'(x) - \Psi_-'(x)\Psi_+(x)$.

Black holes merging, ringdown

- QNMs are observed from the damped oscillations of **gravitational waves** signal in the final - **ringdown** - phase of black holes (BHs) merging.

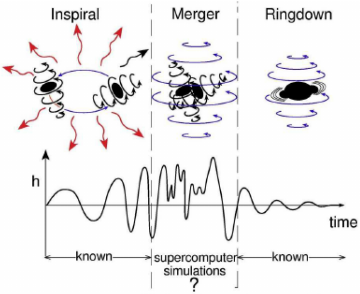


Figure: BH merging (K. Thorne)... OLD PICTURE!

- QNMs can be studied for the **scalar** perturbation, more simply for **Klein-Gordon** equation in curved spacetime.

Importance for modified gravity and phenomenology

- Even if QNMs in GR are well understood, it is still an open problem to compute them in **modified gravity**, so developing **new analytic characterizations and numerical methods** is very important.
- Integrability could become a new powerful mathematical tool in **gravitational phenomenology** and search of new physics.

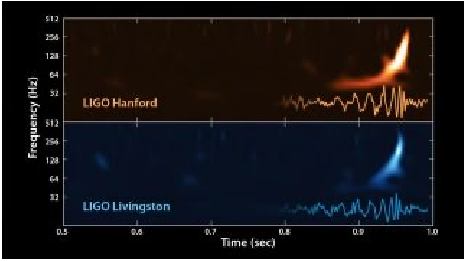


Figure: GW150914 (LIGO)

Alternative models of BHs

- GR BHs present fundamental theoretical problems (e.g. information paradox).
- Also to solve such problems, theoretical models of **Exotic Compact Objects (ECOs)** in alternative theories of gravity have been developed. They have **horizon scale structure**.
- For subtype of ECOs, called **Clean Photosphere Objects (ClePhOs)**, the later stage ringdown signal shows a peculiar train of **echoes**, with **significant deviations from GR**.
- An example of ClePhoS are **fuzzballs** in **String Theory**, with neither horizon nor central singularity and which may solve also the information paradox.

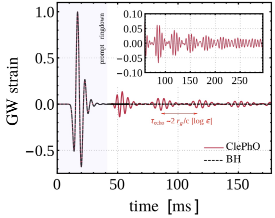


Figure: [Cardoso,Pani:2017]

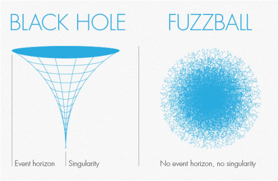


Figure: Olena Shmahalo/Quanta Magazine

Outline

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2. Introduction to ODE/IM
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4. Solving QQ systems via gauge theory
   - $\mathcal{N} = 2$ primer
   - Exact closed formulae for Q
   - On new gauge/gravity duality
   - Comparison of QNMs computations methods
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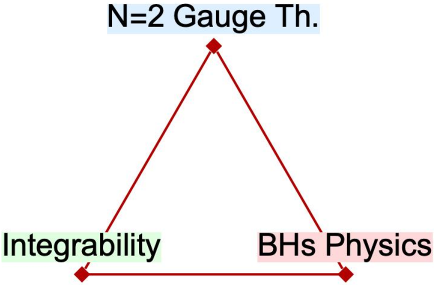
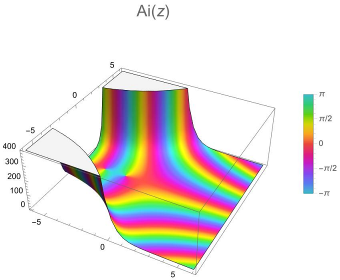


Introduction to ODE/IM

The Stokes phenomenon, Airy function

- An essential ingredient of ODE/IM is the **asymptotic behaviour** of ODE solutions in the **complex domain**, which may vary qualitatively due to Stokes phenomenon.
- For example **Airy ODE**, graph on the right

$$-\frac{d^2}{dx^2}\psi(x) + [x - E]\psi(x) = 0\,. \tag{19}$$



- We can define the **spectral determinant** as

$$Q^{\pm}(E) = \exp[-Z^{\pm}(0)] \prod_{k\ \text{even/odd}} \left(1 - \frac{E}{E_k}\right) \tag{20}$$

and find it satisfies **quantum wronskian** functional relation

$$e^{i\pi/3}Q^{+}(E)Q^{-}(e^{4i\pi/3}E) - e^{-i\pi/3}Q^{+}(e^{4i\pi/3}E)Q^{-}(E) = 2i\,. \tag{21}$$

ODE/IM correspondence essentials

- The first ODE/IM instance found was for the **anharmonic oscillator ODE**, with $\alpha \geq -1$ [Dorey-Tateo,Bazhanov-Lukyanov-Zamolodchikov]

$$-\frac{d^2}{dx^2}\psi(x) + \left(x^{2\alpha} + \frac{l(l+1)}{x^2} - E\right)\psi(x) = 0\,. \tag{22}$$

- The **spectral determinants** are also computed as **Wronskians**

$$Q^+(E,l) = W[\psi_{+,0},\psi_{-,0}] \quad \text{with} \quad \begin{cases} \psi_{+,0}(x) \simeq x^{-\alpha/2}\exp\left\{-\frac{x^{\alpha+1}}{\alpha+1}\right\}, & x \to \infty \\ \psi_{-,0}(x) \simeq x^{l+1}\,, & x \to 0 \end{cases}. \tag{23}$$

- $Q^+(E,l)$ can be **computed** as **integral** of the solution $\mathcal{P}(x) = -i\frac{d}{dx}\ln\psi(x)$ of the **Riccati** ODE equivalent to (22)

$$Q^+(E,l) = \psi_{+,0}(0) = \exp\left\{i\int_0^\infty [\mathcal{P}(x) - \text{regulariz.}]\,dx\right\}. \tag{24}$$

- Through **discrete symmetries** of the ODE, we can define other solutions in $\mathbb{C}$, with $\omega = e^{\frac{i\pi}{\alpha+1}}$

$$\Omega_+ : x \to \omega^{-1}x\,, \quad E \to \omega^2 E \quad \Longrightarrow \quad \psi_{+,k} \equiv \Omega_+^k \psi_{+,0} = \psi_{+,0}(\omega^{-k}x, \omega^{2k}E, l)\,, \qquad (25)$$

so we can write **lateral connection relations** in terms **Stokes multipliers** $T_{k/2}$

$$\psi_{-1}(x) = T_{k/2}(E,l)\psi_{k-1}(x) + \tilde{T}_{k/2}(E,l)\psi_k(x) \quad \Longrightarrow \quad T_{k/2}(E,l) = W[\psi_{-1},\psi_k]\,. \quad (26)$$

- Similarly defining $\psi_{-,1}(x,E,l) \equiv \psi_{-,0}(x,E,-l-1)$ and $Q^-(E,l) = Q^+(E,-l-1)$ from **central connection relations** (between $\psi_{-,k}$ and $\psi_{+,k}$) we obtain QQ **system**

$$\omega^{1/2}Q^+(\omega^{-1}E)Q^-(\omega E) - \omega^{-1/2}Q^+(\omega E)Q^-(\omega^{-1}E) = 2i\,. \qquad (27)$$

- Similarly also TQ and T systems.
- $Q^\pm$, $T_{k/2}$ realize **vacuum** Q, T **functions** of **Scaling 6-vertex IM!**
- Integrals of motions of **KdV/Minimal Models**.

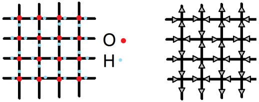


Figure: (Wikicommons/Moali)

Other ODE/IM examples: Liouville

- We can reach the regime $\alpha < -1$, by a formal change of variable, leading to the ODE for the to **Liouville model** , that is CFT central charge $c = 1 + (b + 1/b)^2 \geq 1$

$$-\frac{d^2}{dy^2}\psi(y) + \left[e^{2\theta}(e^{y/b} + e^{-by}) + P^2\right]\psi(y) = 0\,, \tag{28}$$

which reduces to (modified) Mathieu equation for $b = 1$.

- Liouville model is dual to 4D $\mathcal{N} = 2$ susy by **AGT duality**, but we will show through ODE/IM another connection to it!

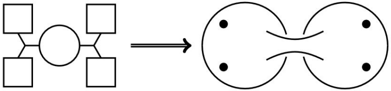


Figure: AGT duality quiver (NLab)

Other ODE/IM examples: Perturbed Hairpin, Paperclip

- **Perturbed Hairpin IM** for $n = 1, 2$ is Doubly Confluent Heun (DCHE)

$$-\frac{d^2}{dx^2}\psi(x) + \left[2kqe^x + k^2(e^{2x} + e^{-nx}) + P^2\right]\psi(x) = 0\,. \tag{29}$$

- **Paperclip IM** for $n = 2$ is the Confluent Heun equation (CHE)

$$-\frac{d^2}{dx^2}\psi(x) + \left[k^2\left(e^x + 1\right)^n - \frac{p^2 e^x}{e^x + 1} - \frac{\left(q^2 - \frac{1}{4}\right)e^x}{\left(e^x + 1\right)^2}\right]\psi(x) = 0\,. \tag{30}$$

- They are both related **2D IQFTs** with **non-conformal boundary interaction**.

Fig. 2. The paperclip formed by a junction of two hairpins.

Figure: Lukyanov-Vitchev-Zamolodchikov

ODE/IM off-critical

- For the quantum Sine-Gordon model we have the coupled ODEs

$$[\partial_z^2 - u(z,\bar{z}) - e^{2\theta} p(z)]\psi = 0 \tag{31}$$
$$[\partial_{\bar{z}}^2 - u(z,\bar{z}) - e^{2\theta} p(\bar{z})]\bar{\psi} = 0 \tag{32}$$

with

$$u = (\partial_z \eta)^2 - \partial_z^2 \eta \qquad \bar{u} = (\partial_{\bar{z}} \eta)^2 - \partial_{\bar{z}}^2 \eta\,, \tag{33}$$

where η solution of the classical **Modified Sinh-Gordon** equation

$$\partial_z \partial_{\bar{z}} \eta - e^{2\eta} + p(z)p(\bar{z})e^{-2\eta} = 0\,, \tag{34}$$

with $p(z) = z^{2\alpha} - s^{2\alpha}$.

ODE/IM correspondence applications

- Over 20+ years, ODE/IM found fruitful applications applications **beyond integrability** in
  - Foundations of quantum mechanics
  - Statistical mechanics
  - Condensed matter physics
  - Supersymmetric gauge theories
  - Pure mathematics

A new very promising application is to **black holes physics**.

On the origin of ODE/IM

- For a long time, ODE/IM has been regarded as either **mysterious correspondence** or just a mathematical coincidence.
- However, new work - arXiv:2106.07600 - has shed light on its **origin**, by "reversing it":

$$\begin{array}{l} QQ/TQ/\text{Bethe Ansatz} \\ \Downarrow \\ \text{Marchenko-like int. eq.} \\ \Downarrow \\ \text{Schrödinger ODEs} \end{array}$$

- This appears general and in principle would provide ODE/IM for also **non-QFTs**, noticebly **spin chains**.

Figure: The Thinker (Rodin)

Integrability for black holes

Heun equations and black holes

- Generally in both **General Relativity** and **modified gravity**, BHs perturbations obey different kind of ODEs
  - Perturbations of **asymptotically AdS$_5$ BHs** obey **Heun** (or higher Fuchsian) equation

$$w''(z) + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\alpha+\beta-\gamma-\delta+1}{z-a}\right) w'(z) + \frac{\alpha\beta z - q}{z(z-1)(z-a)} w(z) = 0\,. \tag{35}$$

  - **Asymptotically flat (Minkowski) BHs** obey **Confluent Heun** equation (CHE)

$$w''(z) + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon\right) w'(z) + \frac{\alpha z - q}{z(z-1)} w(z) = 0\,. \tag{36}$$

  - **Extremal** flat BHs obey **Doubly Confluent Heun** equation (DCHE)

$$w''(z) + \left(\frac{\delta}{z^2} + \frac{\gamma}{z} + 1\right) w'(z) + \frac{\alpha z - q}{z^2} w(z) = 0\,. \tag{37}$$

- Caveat: **rotating BHs** involve **coupled ODEs**!

Heun and Painlevé equations

- **Heun** linear equations can be mapped to **Painlevé** non linear equations
- Painlevé equations similarly are related by a **confluence** process.

$(\underline{1111})$	(standard) Heun	Painlevé VI
$(\underline{11}2)$	confluent Heun	Painlevé nondegenerate V
$(\underline{11}\frac{3}{2})$	degenerate confluent Heun	Painlevé degenerate V
(22)	doubly confluent Heun	Painlevé nondegenerate III′
$(\frac{3}{2}2)$	degenerate doubly confluent Heun	Painlevé degenerate III′
$(\frac{3}{2}\frac{3}{2})$	doubly degenerate doubly confluent Heun	Painlevé doubly degenerate III′
$(\underline{1}3)$	bi-confluent Heun	Painlevé IV
$(\underline{1}\frac{5}{2})$	degenerate bi-confluent Heun	Painlevé 34
(4)	tri-confluent Heun	Painlevé II
$(\frac{7}{2})$	degenerate tri-confluent Heun	Painlevé I

$$\begin{tikzcd} \mathrm{P}_{\mathrm{VI}} \arrow[r] & \mathrm{P}_{\mathrm{V}} \arrow[r] \arrow[d] & \mathrm{P}_{\mathrm{IV}} \arrow[d] & \\ & \mathrm{P}_{\mathrm{III}} \arrow[r] & \mathrm{P}_{\mathrm{II}} \arrow[r] & \mathrm{P}_{\mathrm{I}} \end{tikzcd}$$

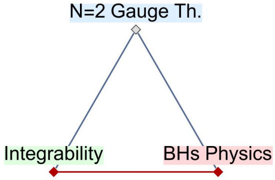
Figure: Heun and Painlevé equations

Quasinormal modes of black holes as Bethe roots

- Through the ODE/IM formula $Q = W[\psi_+, \psi_-]$ we proved that the **Bethe root** (zero) condition on the function

$$Q(\theta_n) = 0\,, \tag{38}$$

is equivalent to the mathematically rigorous definition of **quasinormal modes (QNMs)**!



Derivation: mathematical definition of QNMs vs. ODE/IM correspondence

- By standard techniques (PDE → Laplace tr. → non-hom. ODE →hom. ODE) we can express the perturbation Φ as an expansion over the **quasinormal modes (QNMs)** frequencies ω_n

$$\Phi(t,y) = \sum_n e^{i\omega_n t} \mathrm{Res}\left(\frac{1}{W(s)}\right)\Bigg|_{\omega_n} \int_{-\infty}^{\infty} \psi_-(\omega_n, y_<)\psi_+(\omega_n, y_>)\mathcal{I}(\omega_n, y')\, dy' + \text{ (other contributions)}\,. \tag{39}$$

- QNMs ω_n are so found to be **zeros of wronskian of the fundamental regular solutions** at $y \to \pm\infty$ of the ODE [Nollert:1999], which also defines Q function as

$$W[\psi_+, \psi_-](\omega_n) = 0\,, \qquad \text{with} \quad \psi_\pm(y) \to 0 \quad y \to \pm\infty\,. \tag{40}$$

Y function and quasinormal modes

- One can define a Y **function** as (here for generalized Perturbed Hairpin / generalized extremal RN BHs)

$$Y_{+,+}(\theta) = e^{i\pi(q_1-q_2)} Q_{+,+}(\theta) Q_{-,-}(\theta) \tag{41}$$

- From the QQ system it follows the Y **system** as

$$Y_{+,-}(\theta + \frac{i\pi}{2}) Y_{-,+}(\theta - \frac{i\pi}{2}) = [1 + Y_{+,+}(\theta)][1 + Y_{-,-}(\theta)] \,. \tag{42}$$

- Eventually, the QQ **system** written as

$$e^{i\pi(q_1+q_2)} Q_{+,+}(\theta + \frac{i\pi}{2}) Q_{-,-}(\theta - \frac{i\pi}{2}) = 1 + Y_{+,-}(\theta) \,. \tag{43}$$

  characterizes the **QNMs** with other **quantizations conditions**

$$Y_{+,-}(\theta_n - i\pi/2) = -1 \qquad Y_{-,+}(\theta_n + i\pi/2) = -1 \tag{44}$$

Thermodynamic Bethe Ansatz

- The Y **system can be inverted in the Thermodynamic Bethe Ansatz (TBA)** for $\varepsilon_{\pm,\pm}(\theta) = -\ln Y_{\pm,\pm}(\theta)$:

$$
\begin{aligned}
\varepsilon_{\pm,\pm}(\theta) &= \frac{8\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2} e^{\theta} \mp i\pi(q_1 - q_2) - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{\ln[1+\exp(-\varepsilon_{\pm,\mp}(\theta'))] + \ln[1+\exp(-\varepsilon_{\mp,\pm}(\theta'))]}{\cosh(\theta-\theta')} \\
\varepsilon_{\pm,\mp}(\theta) &= \frac{8\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2} e^{\theta} \mp i\pi(q_1 + q_2) - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{\ln[1+\exp(-\varepsilon_{\pm,\pm}(\theta'))] + \ln[1+\exp(-\varepsilon_{\mp,\mp}(\theta'))]}{\cosh(\theta-\theta')}
\end{aligned}
\tag{45}
$$

- P enters as the $\theta \to -\infty$ boundary condition (from perturbative expansion of ODE) $\varepsilon_{+,+}(\theta, P) \sim 4P\theta + 2C(P, q_1, q_2)$, as $\theta \to -\infty$

Gravity Y system and TBA

- In gravity variables Σ_k (combinations of four charges for generalised RN extremal BHs) the Y system reads

$$
\begin{aligned}
&Y(\theta+\frac{i\pi}{2},-i\Sigma_1,-\Sigma_2,i\Sigma_3,\Sigma_4)Y(\theta-\frac{i\pi}{2},-i\Sigma_1,-\Sigma_2,i\Sigma_3,\Sigma_4)\\
&=[1+Y(\theta,\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4)][1+Y(\theta,-\Sigma_1,\Sigma_2,-\Sigma_3,\Sigma_4)]
\end{aligned}
\tag{46}
$$

- It can be inverted into the TBA

$$
\begin{aligned}
\varepsilon_\pm(\theta)&=[f_{0,+}\mp\frac{i\pi}{2}(\frac{\Sigma_1}{\Sigma_4^{1/4}}-\frac{\Sigma_3}{\Sigma_4^{3/4}})]e^\theta-\varphi*(\bar{L}_\pm+\bar{L}_\mp)(\theta)\\
\bar{\varepsilon}_\pm(\theta)&=[\bar{f}_{0,+}\pm\frac{\pi}{2}(\frac{\Sigma_1}{\Sigma_4^{1/4}}+\frac{\Sigma_3}{\Sigma_4^{3/4}})]e^\theta-\varphi*(L_\pm+L_\mp)(\theta)
\end{aligned}
\tag{47}
$$

where we defined $\varepsilon_\pm(\theta)=-\ln Y(\theta,\pm\Sigma_1,\Sigma_2,\pm\Sigma_3,\Sigma_4)$, $\bar{\varepsilon}_\pm(\theta)=-\ln Y(\theta,\pm i\Sigma_1,-\Sigma_2,\mp i\Sigma_3,\Sigma_4)$, $L=\ln[1+\exp\{-\varepsilon\}]$, $\varphi(\theta)=(2\pi\cosh(\theta))^{-1}$...

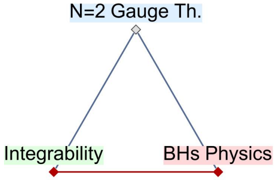
Quasinormal modes from TBA

- The **QNMs condition** in gravity variables **reads alternatively as**

$$\bar{Y}_{+}(\theta_{n'} - i\pi/2) = -1\,, \quad Q_{+,+}(\theta_n) = 0 \tag{48}$$

or

$$\bar{\varepsilon}_{+}(\theta_{n'} - i\pi/2) = -i\pi(2n' + 1)\,. \tag{49}$$



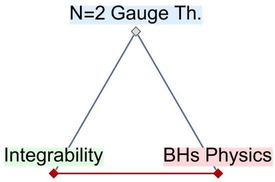
- Through the quantization condition on $\bar{\varepsilon}_{+}$ we can **actually numerically compute QNMs from TBA**.

n	l	TBA	Leaver	WKB
0	1	$0.869623 - 0.372022i$	$0.868932 - 0.372859i$	$0.89642 - 0.36596i$
0	2	$1.477990 - 0.368144i$	$1.477888 - 0.368240i$	$1.4940 - 0.36596i$
0	3	$2.080200 - 0.367076i$	$2.080168 - 0.367097i$	$2.0916 - 0.36596i$
0	4	$2.680363 - 0.366637i$	$2.680350 - 0.366642i$	$2.6893 - 0.36596i$

Table: Comparison of QNMs in different methods ($n' = 0$, $\Sigma_1 = \Sigma_3 = 0.2$, $\Sigma_2 = 0.4$, $\Sigma_4 = 1$).

Other applications of integrability

- We notice that **much of the BH theory seems to go in parallel to the ODE/IM correspondence** construction and its 2D integrable field theory interpretation, beyond the determination of QNMs!
  - For instance, also the **greybody factor** that parametrizes the **Hawking radiation** seems to be **ratio of Qs**
  - Intepretation and use of T, TQ relation...



Details: Alternative QNMs quantizations condition for D3 brane

- From the relation between T and a it follows a proof of the **alternative quantization condition for quasinormal modes** (only for the $N_f = 0$ $SU(2)$ theory, that is the D3 brane)

$$a(\hbar, \Lambda_0) = n/2\,, \qquad n \in \mathbb{Z}\,. \tag{50}$$

- Indeed, the TQ relation $T(\theta)Q(\theta) = Q(\theta - i\pi/2) + Q(\theta + i\pi/2)$ means $Q(\theta_n - i\pi/2) + Q(\theta_n + i\pi/2) = 0$. This and the QQ relation $Q(\theta + i\pi/2)Q(\theta - i\pi/2) = 1 + Q^2(\theta)$ actually fixes $Q(\theta_n + i\pi/2)Q(\theta_n - i\pi/2) = 1$ and then

$$Q(\theta_n \pm i\pi/2) = \pm i \tag{51}$$

are fixed, too. Again the QQ relation around θ_n forces $Q(\theta + i\pi/2) = i \pm Q(\theta) + \ldots$ and $Q(\theta - i\pi/2) = -i \pm Q(\theta) + \ldots$ up to smaller corrections (dots). Therefore, TQ relation imposes

$$T(\theta_n) = 2\cos 2\pi a(\theta_n) = \pm 2\,. \tag{52}$$

Solving QQ systems via gauge theory

Summary of $\mathcal{N} = 2$ gauge (Seiberg-Witten) theory

- Seiberg-Witten (SW) theory is the **effective theory** for $\mathcal{N} = 2$ gauge theory.
- Its lagrangian is expressed in terms of a **SW prepotential** $\mathcal{F}_{SW}$

$$\mathcal{L} = \frac{1}{16\pi}\mathrm{Im}\int d^4x \left[\frac{1}{2}\int d^2\theta\, \mathcal{F}''_{SW}(\phi) W^\alpha W_\alpha + \int d^2\theta d^2\bar{\theta}\phi^\dagger \mathcal{F}'_{SW}(\phi)\right] . \tag{53}$$

- $\mathcal{F}_{SW}$ can be obtained from **classical SW periods** $a^{(0)}$, $a_D^{(0)}$

$$a_D^{(0)} = \frac{\partial \mathcal{F}_{SW}}{\partial a^{(0)}} , \tag{54}$$

$$a^{(0)}(u, \mathbf{m}, \Lambda) = \oint_{\mathcal{A}} \lambda(x, u, \mathbf{m}, \Lambda)\, d\, x , \qquad a_D^{(0)}(u, \mathbf{m}, \Lambda) = \oint_{\mathcal{B}} \lambda(x, u, \mathbf{m}, \Lambda)\, d\, x , \tag{55}$$

which are integrals of a SW differential λ related to the SW hyperelliptic curve

$$\lambda \sim \sqrt{x^3 + c_2x^2 + c_1x + c_0} . \tag{56}$$

Quantum Seiberg-Witten theory

- To compute **instanton contributions** spacetime is deformed by two complex parameters ϵ_1, ϵ_2 into the Ω-**background**.
- More easily connected to integrability and gravity is the **Nekrasov-Shatashvili (NS) limit** $\epsilon_2 \to 0, \epsilon_1 = \hbar \neq 0$, in which the **quantum SW curves** are just ODEs in $y \sim -\ln x$

$$- \hbar^2 \frac{d^2}{dy^2} \psi(y) + \left[U(y; \mathbf{m}, \Lambda) - u \right] \psi(y) = 0 \,. \tag{57}$$

- The **NS prepotential** $\mathcal{F}$ has as leading $\hbar \to 0$ order the SW prepotential

$$\mathcal{F} = \frac{1}{\hbar} \mathcal{F}_{SW} + O(\hbar) + O(e^{-\frac{1}{\hbar}}) \qquad \hbar \to 0 \,. \tag{58}$$

Nekrasov instanton series

- The NS prepotential can be obtained as a **series in the instanton parameter** Λ

$$\mathcal{F}(\Lambda, a) = \mathcal{F}^{pert}(a) + \sum_{n=1}^{\infty} \mathcal{F}_n^{inst}(a) \Lambda_{N_f}^{(4-N_f)n} \tag{59}$$

- The **gauge period** a can be also expressed in terms of u through Matone's relation

$$u = a^2 + \frac{2\Lambda_{N_f}}{4 - N_f} \frac{\partial \mathcal{F}_{inst}(a, \Lambda)}{\partial \Lambda_{N_f}} \tag{60}$$

- Also the **gauge dual instanton period** A_D is defined as

$$A_D = \frac{\partial \mathcal{F}}{\partial a} \tag{61}$$

and for $\hbar \to 0$, $A_D^{(0)} \simeq a_D^{(0)}$, but in general A_D and ODE **monodromy dual period** a_D differ non-perturbatively.

Example of instanton series

- For example for $N_f = 3$ at second order

$$
\begin{aligned}
\mathcal{F}_{inst}(\Lambda_3, m_1, m_2, m_3, a) = &-\frac{m_1 m_2 m_3}{4(4a^2-\hbar^2)}\Lambda_3 \\
&+\left[\frac{-256a^8-256a^6\left(m_3^2+m_1^2+m_2^2-2\hbar^2\right)+96a^4\left(2m_3^2\hbar^2+2m_2^2\left(4m_3^2+\hbar^2\right)+2m_1^2\left(4m_3^2+4m_2^2+\hbar^2\right)-3\hbar^4\right)}{8192\left(a^2-\hbar^2\right)\left(4a^2-\hbar^2\right)^3}\right. \\
&+\frac{-16a^2\left(3m_3^2\hbar^4+3m_2^2\left(8m_3^2\hbar^2+\hbar^4\right)+m_1^2\left(8m_2^2\left(10m_3^2+3\hbar^2\right)+3\left(8m_3^2\hbar^2+\hbar^4\right)\right)-4\hbar^6\right)}{8192\left(a^2-\hbar^2\right)\left(4a^2-\hbar^2\right)^3} \\
&\left.+\frac{4m_3^2\hbar^6+4m_2^2\hbar^6+48m_3^2m_2^2\hbar^4+4m_1^2\hbar^2\left(12m_3^2\hbar^2-4m_2^2\left(28m_3^2-3\hbar^2\right)+\hbar^4\right)-5\hbar^8}{8192\left(a^2-\hbar^2\right)\left(4a^2-\hbar^2\right)^3}\right]\Lambda_3^2+O(\Lambda_3^3)\,. \qquad (62)
\end{aligned}
$$

Nice!...But...
the terms are infinite and the computation time for higher orders grows (roughly) exponentially.

No jokes : Schwarshild black hole !

- ODE (30) for **Paperclip IM generalizes** to that for $N_f = 3$ $SU(2)$ **gauge th.**

$$-\hbar^2 \frac{d^2}{dy^2}\psi(y) + \Big\{ e^{2y}\Lambda_3 \left(4(m_1-m_2)^2\right) + 4e^y\sqrt{\Lambda_3}\left(-2\hbar^2 + 8m_1m_2 + \Lambda_3 m_3 - 8u\right) \\ + \left(\Lambda_3^2 - 24\Lambda_3 m_3 + 64u\right) + 4e^{-y}\sqrt{\Lambda_3}\left(8m_3 - \Lambda_3\right) + 4\Lambda_3 e^{-2y} \Big\} \frac{1}{16\left(\sqrt{\Lambda_3}e^y - 2\right)^2}\psi(y) = 0\,. \tag{63}$$

- Also through the following map

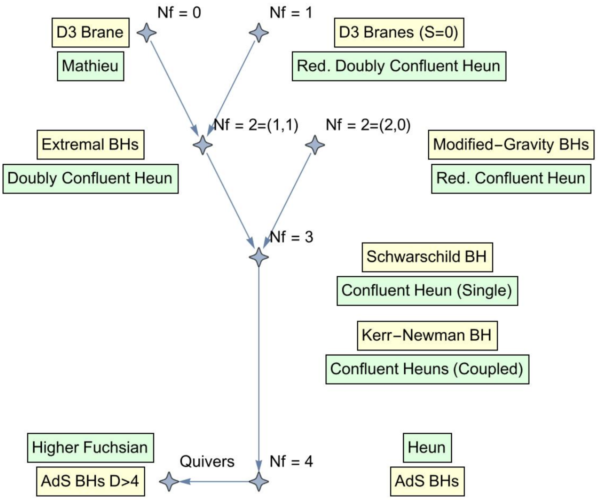
$$r = \frac{4M}{\Lambda_3}e^{-y}\,, \qquad \hbar = 1\,, \qquad \Lambda_3 = -16iM\omega\,, \qquad u = l(l+1) - 8M^2\omega^2 + \frac{1}{4}\,, \\ m_1 = s - 2iM\omega\,, \qquad m_2 = -s - 2iM\omega\,, \qquad m_3 = -2iM\omega\,, \tag{64}$$

becomes the Regge-Wheeler equation for the **Schwarschild BH !**

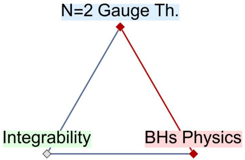
$$\left[f(r)\frac{d}{dr}f(r)\frac{d}{dr} + \omega^2 - f(r)\left[\frac{l(l+1)}{r^2} + (1-s^2)\frac{2M}{r^3}\right]\right]\phi(r) = 0\,, \quad \text{with } f(r) = \left(1 - \frac{2M}{r}\right) \tag{65}$$

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

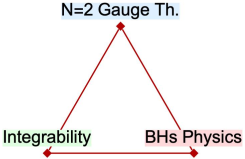
General picture of new gauge-gravity-ODE/IM triality



- $SU(2)$ $\mathcal{N}=2$ (NS) gauge theories and BHs models.



- We extended the triality with ODE/IM until $N_f = 3$ (single).



Solving QQ systems: D3 brane

- The scalar perturbation on the **D3 brane**, obeys Mathieu equation, a particular case of DCHE, which is the same as for self-dual Liouville IM and $N_f = 0$ quantum SW curve.
- Through linear connection relations among solutions ψ_k one easily derives QQ-systems like

$$Q(\theta + i\pi/2, a)Q(\theta - i\pi/2, a) - Q(\theta, a)^2 = 1 \,. \tag{66}$$

- From the QQ **system** follows a **closed formula for** Q

$$Q(\theta, a) = \frac{\sinh\left[\frac{1}{\hbar}A_D(\theta, a)\right]}{\sin\left[\frac{2\pi}{\hbar}a\right]} \,, \quad \text{with} \quad A_D\,(\theta + i\pi/2, a) = A_D(\theta, a) - 2\pi i a \,, \tag{67}$$

We proved this and the following formulae purely from **ODE/IM and gauge theory**, but **alternative derivation** is possible through **AGT duality**

Solving QQ systems: extremal black holes

- We proved the QQ system for $N_f = 2$ gauge theory, that is **extremal BHs**

$$Q\left(\theta + i\pi/2, -m_1, m_2\right) Q\left(\theta - i\pi/2, m_1, -m_2\right) - Q(\theta, -m_1, -m_2) Q(\theta, m_1, m_2) = e^{\frac{i\pi}{\hbar}(m_1 - m_2)} \tag{68}$$

is equivalent to the shift relations

$$A_D\left(\theta \pm i\pi/2, m_1, -m_2, a\right) = A_D\left(\theta, m_1, m_2, a\right) + h\left(m_2, a\right) \mp i\pi a \tag{69}$$
$$(-\partial_{m_1} - \partial_{m_2})\mathcal{F}_{inst}(\theta + i\pi/2, -m_1, -m_2, a) + (\partial_{m_1} + \partial_{m_2})\mathcal{F}_{inst}(\theta - i\pi/2, m_1, m_2, a) = 0$$

and similar others, with $h(m, a) = \ln \sqrt{\cos \frac{\pi}{\hbar}(a - m) \sec \frac{\pi}{\hbar}(a + m)}$.

- The **exact closed formula** for Q for $N_f = 2$ ensues

$$Q(\theta, m_1, m_2, a) = 2\pi \exp\left\{\frac{1}{\hbar}\sum_{k=1}^{2} \frac{\partial\mathcal{F}\left(\theta, m_1, m_2, a\right)}{\partial m_k}\right\} \frac{\sinh \frac{1}{\hbar} A_D\left(\theta, m_1, m_2, a\right)}{\sin \frac{2\pi}{\hbar} a} . \tag{70}$$

Solving QQ systems: general (Minkowski) black holes (I)

- We proved the first QQ system for $N_f = 3$ gauge theory, that is **Schwarschild BHs**

$$Q_1(\theta + i\pi, m_1, m_2, -m_3)Q_1(\theta, -m_1, -m_2, m_3) - Q_1(\theta, m_1, m_2, m_3)Q_1(\theta + i\pi, -m_1, -m_2, -m_3) = \frac{2i}{\hbar}(m_1 + m_2)\,, \tag{71}$$

is equivalent to the shift relations

$$A_D\,(\theta \pm i\pi, m_1, m_2, -m_3, a) = A_D\,(\theta, m_1, m_2, m_3, a) + h\,(m_3, a) \mp i\pi a\,, \tag{72}$$

$$(\partial_{m_1} + \partial_{m_2} - \partial_{m_3})\mathcal{F}_{inst}(\theta + i\pi, m_1, m_2, -m_3, a) + (-\partial_{m_1} - \partial_{m_2} + \partial_{m_3})\mathcal{F}_{inst}(\theta, -m_1, -m_2, m_3, a) = 0$$

and similar others.

- The **exact closed formula** for Q_1 ensues

$$Q_1\,(\theta, \mathbf{m}, a) = \tag{73}$$

$$\pi i 2^{\frac{3}{2}} e^{\frac{1}{\hbar}\left[m_3\theta + \left(\frac{m_1+m_2}{2} + m_3 - \frac{\Lambda_3}{16}\right)\ln 2\right]}\Gamma\!\left(1 + \frac{m_1 + m_2}{\hbar}\right)\exp\left\{\frac{1}{\hbar}\sum_{k=1}^{3}\frac{\partial\mathcal{F}\,(\theta, \mathbf{m}, a)}{\partial m_k}\right\}\frac{\sinh\frac{1}{\hbar}A_D\,(\theta, \mathbf{m}, a)}{\sin\frac{2\pi}{\hbar}a}\,.$$

Solving QQ systems: general (Minkowski) black holes (II)

- We proved the second QQ system for $N_f = 3$ gauge theory, that is **Schwarschild BHs**

$$Q_2(\theta, -q_2, -q_1, q_3)Q_2(\theta, q_2, q_1, q_3) - Q_2(\theta, q_1, q_2, q_3)Q_2(\theta, -q_1, -q_2, q_3) = q_1^2 - q_2^2 \tag{74}$$

  is equivalent to the shift relation

$$\partial_{m_1}\mathcal{F}_{inst}(\theta, m_1, m_2, m_3, a) - \partial_{m_1}\mathcal{F}_{inst}(\theta, -m_1, -m_2, m_3, a) = 0\,. \tag{75}$$

  and similar others

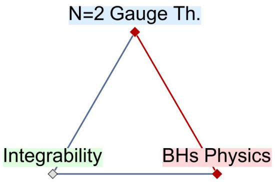
- The **exact closed formula** for Q_2 ensues

$$Q_2\left(\theta, \mathbf{m}, a\right) = 2^{\frac{m_2}{\hbar}}\Gamma(1+\frac{m_1+m_2}{\hbar})\Gamma(1+\frac{m_1-m_2}{\hbar})\exp\left\{\frac{2}{\hbar}\frac{\partial\mathcal{F}\left(\theta, \mathbf{m}, a\right)}{\partial m_1}\right\}\,. \tag{76}$$

On the new ($\mathcal{N} = 2$) gauge / gravity duality

- Since proved a **relation between the Q function and the gauge periods**

$$Q(\theta, \mathbf{m}, a) = \mathfrak{F}(\theta, \mathbf{m}, a) \frac{\sinh \frac{1}{\hbar} A_D(\theta, \mathbf{m}, a)}{\sin \frac{2\pi}{\hbar} a} \,, \tag{77}$$



from our formalism **it follows immediately that QNMs are given by quantization conditions on the gauge periods**

$$Q(\theta, \mathbf{m}, a) = 0 \quad \Longrightarrow \quad A_D(\theta, \mathbf{m}, a) = 2i\pi\hbar\, n \quad \text{for } n \in \mathbb{Z} \,. \tag{78}$$

- (78) was previously found heuristically and allowed:
  - a **novel analytic characterization of QNMs**;
  - to **find new results for the BHs theory** (by many other authors);
  - an **unexpected application of Supersymmetry**.

A fruitful new field

The importance of this result is manifold.

1. It constitutes a **novel analytic characterization of QNMs**, for which previously very few were known [arXiv:2006.06111].
2. In the increasingly growing outflow of research on this topic, it has already allowed to **find new results** for the BHs theory, such as:
    1. an **isospectral simpler equation** to the perturbation ODE [arXiv:2007.07906];
    2. improved **theoretical proofs of BHs stability** [arXiv:2105.13329];
    3. **more accurate computations** of observable quantities such as **Love numbers**, describing tidal deformations [arXiv:2105.04483];
    4. a **simpler interpretation of Chandrasekhar transformation** as exchange of gauge mass parameters [arXiv:2111.05857];
    5. precise determination of the **conditions of invariance under (Couch-Torrence) transformations** which exchange inner horizon and null infinity [arXiv:2203.14900];
    6. an **exact formula** for the thermal scalar **two-point function** in four dimensional **holographic CFT** [arXiv:2206.07720].
    7. ...

Comparison of QNMs computations methods

n	l	TBA	Leaver	WKB
0	1	$0.896681 - 0.40069i$	N.A.	$0.93069 - 0.39458i$
0	2	$1.5308 - 0.39676i$	N.A.	$1.5511 - 0.39458i$
0	3	$2.15708 - 0.395689i$	N.A.	$2.1716 - 0.39458i$
0	4	$2.78077 - 0.39525i$	N.A.	$2.7921 - 0.39458i$

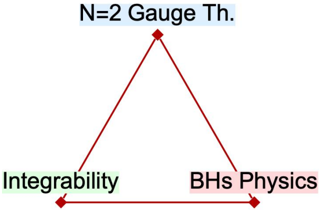
Table: Here $\Sigma_1 \neq \Sigma_3$ and the (original) Leaver method seems not applicable (N.A.).

- Computing QNMs exactly is a non-trivial mathematical problem, it has been typically quite **laborious**, also because of their **few exact analytic characterizations**.
- The standard analytic method is the one with the **continued fractions** by Leaver is laborious and we have found it **sometimes not applicable** (in its original form).
- Application of $\mathcal{N} = 2$ **gauge theory** is a new analytic characterization of QNMs, but it requires a **nontrivial re-summation procedure** for $\omega_n \sim \Lambda_n \sim 1$.
- Our **integrability** exact method is **direct and simple**, but now we have developed for just a **few models** (see below).

Conclusions

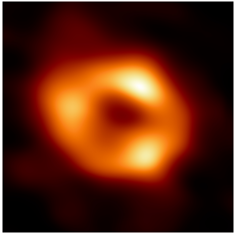
Summary

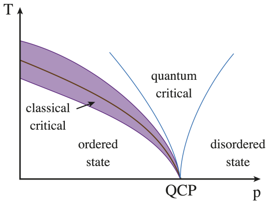
- We have shown how **integrable models** when studied in **ODE/IM correspondence** approach can find:
  - natural connection to (deformed) $\mathcal{N} = 2$ **SUSY gauge theory**
  - as well as to **black hole perturbation theory**
  - and shed light on the **relation** recently found **between the two**.
- This **new triality** besides being interesting in itself allows to:
  - **find new results** an all three sides of the correspondence
  - at the **non-perturbative exact level**.
- Especially for QNMs of BHs we give:
  - **new analytic characterizations** from Q, Y, T functions
  - **new method of computation** as Thermodyn. Bethe Ansatz.



Future developments

- $\mathcal{N} = 2$ gauge theories and BHs constitute **novel fields of application of integrability** many future developments could be pursued:
    - **model generalization**;
    - **other BHs observables**;
    - **non-linear** perturbation theory;
    - **connection to other fields** (like holography).





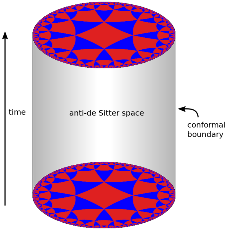
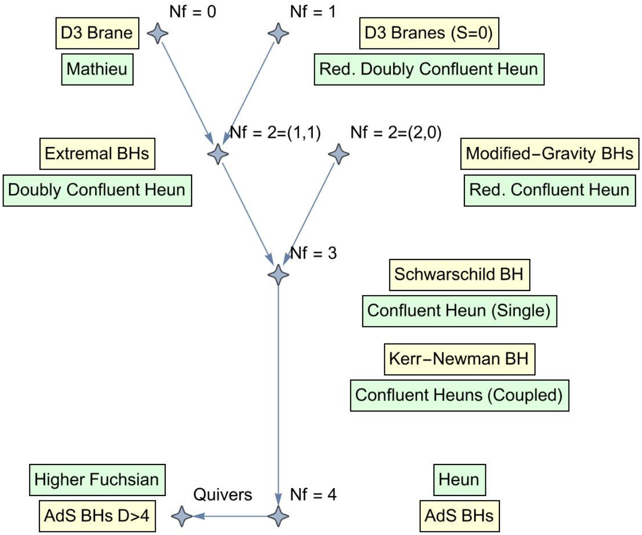


Figure: (Wikicommons/Herzog,Dunkel)



Thank you

for your attention!