ODE/IM correspondence applications

to $\mathcal{N} = 2$ gauge theory and black holes physics

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ODE/IM for Heun equations

- ODE/IM allows to apply integrability to other different physical theories with same ODE.
- In general we consider the following ODEs
  - Doubly Confluent Heun (2 irregular sings.)

$$\frac{d^2\psi}{dz^2} + \left(\frac{\delta}{z^2} + \frac{\gamma}{z} + 1\right)\frac{d\psi}{dz} + \frac{\alpha z - q}{z^2}\psi = 0 \tag{1}$$

  - Confluent Heun (1 irr., 2 reg. sings.)

$$\frac{d^2\psi}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon\right)\frac{d\psi}{dz} + \frac{\alpha z - q}{z(z-1)}\psi = 0 \tag{2}$$

- The Q function of some Integrable Model (IM) can be defined as wronskian of regular solutions at different singular points

$$Q_{jk} = W[\psi_j, \psi_k], \quad \psi_{j,k} \to 0, \; z \to z_j, z_k \tag{3}$$

- One can derive also T, Y functions as well as their functional (QQ, TQ, Y system) and integral equations (TBA) they satisfy.
- Stokes sectors clearer mapping $z \sim e^y$, as for self-dual Liouville IM (Mathieu ODE) [1]

$$-\frac{d^2}{dy^2}\psi + \left[2e^{2\theta}\cosh y + p^2\right]\psi = 0\,. \tag{4}$$

Integrability & N=2 gauge th.

- The Heun confluence process corresponds to the renormalization flow through $SU(2)$ $\mathcal{N} = 2$ Nekrasov-Shatashvili (NS) gauge theories

$$\begin{array}{ll} N_f = 4 \to N_f = 3 & \to N_f = 2 || N_f = 1 || N_f = 0 \\ \text{Heun} \to \text{Conf. H.} & \to \text{Doub. Conf. H.} \end{array} \tag{5}$$

- Using the ODE/IM correspondence we connected basic integrability Q, T and Y functions to the quantum gauge theory periods a, a_D. For instance (for self-dual Liouville IM and $N_f = 0$ gauge theory) we proved: [1]

$$\begin{aligned} Q &= \exp[2\pi i a_D] \\ T &= 2\cos[2\pi a]\,, \end{aligned} \tag{6}$$

- This allowed also to find several new results on both sides, for instance:
  1. an exact non linear integral equation (Thermodynamic Bethe Ansatz, TBA) for the gauge periods;
  2. an interpretation of the integrability functional relations as new exact R-symmetry relations for the periods.

  In details, the TQ relation

$$T(\theta, a)Q(\theta, a) = Q(\theta - i\pi/2, a) + Q(\theta + i\pi/2, a) \tag{7}$$

  turns out to be a new exact R-symmetry relation for the periods

$$a_D(\theta + i\pi/2, a) = -ia_D(\theta, a) - i\hbar \ln\Big\{-\cos[2\pi a] - \sqrt{\cos^2[2\pi a] - 1 - \exp[-4\pi i a_D(\theta, a)]}\Big\}\,. \tag{8}$$

- This is a new kind of gauge/integrability correspondence and we developed it until now for the $SU(2)$ $N_f = 0, 1, 2$ [1, 4] and $SU(3)$ $N_f = 0$ $\mathcal{N} = 2$ [2] (NS deformed) gauge theories, in correspondence with Liouville, Pertubed Hairpin, A_2 Toda IMs.

Quasinormal modes of BHs from Thermodynamic Bethe Ansatz

- Perturbations Φ of black holes (BHs) are mathematically solutions of some PDE (e.g. linearized Einstein, Klein Gordon), which expand in quasinormal modes (QNMs) ω_n as

$$\Phi(t, x) = \sum_n e^{i\omega_n t}\,\mathrm{Res}\left(\frac{1}{W(\omega)}\right)\Bigg|_{\omega_n} \int_{-\infty}^{\infty} \psi_-(\omega_n, x_<)\psi_+(\omega_n, x_>)\mathcal{I}(\omega_n, x')\,dx'\,, \tag{9}$$

  where x is the tortoise coordinate, $\mathcal{I}$ is given by initial data on the perturbation and $\psi_\pm$ are solutions of the ODE associated to the PDE through Laplace transform.
- Even if QNMs in GR are well understood, it is still an open problem to compute them in modified gravity, so developing new analytic characterizations and numerical methods is very important, also for gravitational waves observations (ringdown phase) and search of new physics [6].
- Crucially, we noticed that QNMs are given by the zeros of the same wronksian which in ODE/IM correspondence defines the Baxter's Q function of some IM [3]

$$W(\omega_n) = W[\psi_+, \psi_-](\omega_n) = Q(\omega_n) = 0\,. \tag{10}$$

  In particular, ODE (1) is for the scalar perturbation of the extremal 4 charges BH (intersection of D3 branes)

$$ds^2 = -f(r)dt^2 + f(r)^{-1}[dr^2 + r^2 d\Omega_2^2]\,, \quad f(r) = \prod_{i=1}^{4}(1 + \mathcal{Q}_i/r)^{-\frac{1}{2}} \tag{11}$$

  under the change of parameters, with $\Sigma_k = \sum_{i_1<\ldots<i_k}^4 \mathcal{Q}_{i_1}\cdots\mathcal{Q}_{i_k}$

$$\begin{aligned} &\delta = 4e^{2\theta} = -4\omega^2\sqrt{\Sigma_4}\,, && \gamma = 2 + 2e^{\theta}\Sigma_3/\Sigma_4^{3/4}, \\ &\alpha = 1 - e^{\theta}\Sigma_1/(2\sqrt[4]{\Sigma_4}) + e^{\theta}\Sigma_3/(2\Sigma_4^{3/4}), && q = -2e^{2\theta} - e^{2\theta}\Sigma_3^2/(4\Sigma_4^{3/2}) - e^{\theta}\Sigma_3/(2\Sigma_4^{3/4}) + l(l+1) - \omega^2\Sigma_2\,. \end{aligned} \tag{12}$$

- It follows also a new effective method to exactly compute (numerically) the quasinormal modes: the Thermodynamic Bethe Ansatz (TBA) for $\varepsilon = -\ln Y$, (with $\varphi = 1/\cosh\theta$, $L = \ln[1 + \exp(-\varepsilon)]$)

$$\begin{aligned} \varepsilon_{\pm,\pm}(\theta) &= [f_{0,+} \mp i\pi(\Sigma_1/(2\sqrt[4]{\Sigma_4}) - \Sigma_3/(2\Sigma_4^{3/4})]e^{\theta} - \varphi * (\bar{L}_{\pm\pm} + \bar{L}_{\mp\mp})(\theta) \\ \bar{\varepsilon}_{\pm,\pm}(\theta) &= [\bar{f}_{0,+} \pm \pi(\Sigma_1/(2\sqrt[4]{\Sigma_4}) + \Sigma_3/(2\Sigma_4^{3/4})]e^{\theta} - \varphi * (L_{\pm\pm} + L_{\mp\mp})(\theta)\,, \end{aligned} \tag{13}$$

  which we compare with other standard methods and sometimes find convenient.

n	l	TBA	Continued Fractions	WKB approximation
0	1	$0.869623 - 0.372022i$	$0.868932 - 0.372859i$	$0.89642 - 0.36596i$
0	2	$1.477990 - 0.368144i$	$1.477888 - 0.368240i$	$1.4940 - 0.36596i$
0	3	$2.080200 - 0.367076i$	$2.080168 - 0.367097i$	$2.0916 - 0.36596i$
0	4	$2.680363 - 0.366637i$	$2.680350 - 0.366642i$	$2.6893 - 0.36596i$

- The spacetimes for which we show this are in the simplest $SU(2)$ $N_f = 0$ case the D3 brane, in the $N_f = 1, 2$ case a generalization of extremal Reissner-Nordström (charged) BHs [3, 4] and more recently in the $N_f = 3$ case also asymptotically flat Schwarschild BHs [5, 8].

ODE/IM vs AGT duality

- Through linear connection relations among solutions ψ_k one easily derives QQ-systems like (for Liouville IM or $N_f = 0$ gauge theory):

$$Q(\theta + i\pi/2, a)Q(\theta - i\pi/2, a) - Q(\theta, a)^2 = 1\,. \tag{14}$$

- The QQ system is solved formally by

$$Q(\theta, a) = \frac{\sinh[2\pi A_D(\theta, a)]}{\sin[2\pi a]}\,, \quad \text{with} \tag{15}$$

$$A_D(\theta + i\pi/2, a) = A_D(\theta, a) - ia\,. \tag{16}$$

- a and A_D are $\mathcal{N} = 2$ gauge periods computable from NS prepotential instanton expansion

$$\begin{cases} A_D &= \partial\mathcal{F}/\partial a \\ a &= \partial\mathcal{F}/\partial\Lambda_0 \end{cases} \qquad \mathcal{F} = \mathcal{F}^{pert} + \sum_{n=1}^{\infty}\mathcal{F}_n^{inst}\Lambda_0^{4n} \tag{17}$$

- Analytic expressions for a few orders ensue, while TBA gives numerical resummation.
- An alternative derivation is possible through AGT duality, but ODE/IM suffices.
- We derived similar results for $N_f = 2, 3$ [8].

Possible future developments

- Model extension like for Kerr, modified gravity BHs, fuzzballs, with coupled ODEs.
- Applications of integrability beyond QNMs (*e.g.* the *greybody factor* seems ratio of Qs.).
- Extending to Heun equation, corresponding to AdS BHs (or $SU(2)$ $N_f = 4$ gauge theory) would allow further connection with holographic duality and its applications like hydrodynamics and phase transitions.

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