Integrability methods in $\mathcal{N} = 2$ gauge theory and black holes perturbation theory

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On quantum integrability

- Broadly speaking, integrability can be considered as the study of **non-linear** phenomena in nature in a quantitative **exact way (non perturbative)**.
- Integrable structures appear also in **4D SUSY gauge theories**, typically with $\mathcal{N} = 4$ supersymmetry (AdS/CFT correspondence) but also with $\mathcal{N} = 2$.
- The hallmark of quantum integrability is the presence of **infinite (local) integrals of motion commuting** with each other

$$[\mathbf{I}_{2n-1}, \mathbf{I}_{2m-1}] = 0\,, \tag{1}$$

which are also asymptotic expansion coefficients of the **Baxter's** Q **operator**

$$\ln \mathbf{Q}(\theta) \simeq -C_0 e^{\theta} - \sum_{n=1}^{\infty} e^{\theta(1-2n)} C_n \mathbf{I}_{2n-1} \qquad \theta \to +\infty\,. \tag{2}$$

The ODE/IM correspondence

- The **ODE/IM correspondence** is an approach to integrability [arXiv:9812211, 9812247, 9906219], in which the Q **function** (vacuum eigenvalue of Q operator) is defined as the **wronskian** of the regular solutions at different singular points

$$Q = W[\psi_+, \psi_-] \qquad \text{with} \qquad \psi_\pm(y) \to 0 \quad y \to \pm\infty \tag{3}$$

of some ODE, like (for self-dual Liouville IM):

$$-\frac{d^2}{dy^2}\psi + \left[2e^{2\theta}\cosh y + P^2\right]\psi = 0\,. \tag{4}$$

- From ODE's (lateral/radial) connection relations, one can derive also T, Y **functions** as well as the **functional and integral equations** they satisfy.
- The ODE/IM is an **elegant approach** to integrability which **allows to apply it to very different physical theories!**

Three different physical theories, same mathematics!

$$-\frac{d^2}{dy^2}\psi(y) + \left[2e^{2\theta}\cosh y + P^2\right]\psi(y) = 0 \qquad \text{INTEGRABILITY (sd-Liouville)} \tag{5}$$

$$\frac{\hbar}{\Lambda_0} = \frac{\epsilon_1}{\Lambda_0} = e^{-\theta} \qquad \frac{u}{\Lambda_0^2} = \frac{1}{2}P^2 e^{-2\theta} \qquad \Big\Downarrow \tag{6}$$

$$-\frac{\hbar^2}{2}\frac{d^2}{dy^2}\psi(y) + [\Lambda^2\cosh y + u]\psi(y) = 0 \qquad \text{N=2 GAUGE TH. } (SU(2)\ N_f = 0) \tag{7}$$

$$r = Le^{y/2} \quad \omega L = -2ie^{\theta} \quad P = \frac{1}{2}(l+2) \qquad \Big\Downarrow \qquad \phi(r) = e^{y/2}\psi(y) \tag{8}$$

$$\frac{d^2\phi}{dr^2} + \left[\omega^2\left(1+\frac{L^4}{r^4}\right) - \frac{(l+2)^2 - \frac{1}{4}}{r^2}\right]\phi(r) = 0 \qquad \text{BLACK HOLES PERT. (D3 brane)} \tag{9}$$

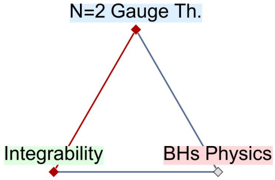
- We have shown this **triality** generally for the Doubly Confluent Heun eq. - **extremal BHs** - but work in progress also on Confluent Heun eq. - general **asymptotically flat BHs**.

A new ($\mathcal{N} = 2$) gauge / integrability correspondence

- Using ODE/IM correspondence, we **connected** the **basic integrability** Q**,** Y **and** T **functions** to the **gauge exact quantum periods** a, a_D (building blocks of the $\mathcal{N} = 2$ **gauge prepotential**), proving relations like (here for self-dual Liouville and $SU(2)$ $N_f = 0$):

$$Q(\theta, P) = \exp\left[\frac{2\pi i}{\hbar} a_D(\hbar, u, \Lambda_0)\right] \qquad T(\theta, P) = 2\cos\left[\frac{2\pi}{\hbar} a(\hbar, u, \Lambda_0)\right], \tag{10}$$

under the parameters map $\frac{\hbar}{\Lambda_0} = \frac{\epsilon_1}{\Lambda_0} = e^{-\theta}$, $\frac{u}{\Lambda_0^2} = \frac{1}{2}P^2 e^{-2\theta}$.

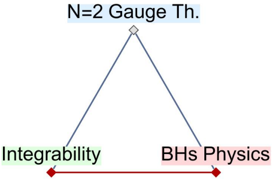


- This basic identification allowed us to find several **new results for both sides**, for instance:
  1. an interpretation of the integrability functional relations as **new exact** R**-symmetry relations for the periods**;
  2. an exact non linear integral equation (**Thermodynamic Bethe Ansatz, TBA**) for the **gauge (dual) periods**;
  3. **new formulas for the local integrals of motion** in terms of gauge periods
  4. **exact computation of the** $\mathcal{N} = 2$ **gauge prepotential** $\mathcal{F}_{NS}$ (effective low energy theory)

Quasinormal modes as Bethe roots of integrability

- We proved that the **quasinormal modes (QNMs) condition** is a **Bethe root** (zero) condition on the $Q = W[\psi_+, \psi_-]$ function

$$Q(\theta_n) = 0\,. \tag{11}$$



Derivation: mathematical definition of QNMs vs. ODE/IM correspondence

- By standard techniques (PDE $\to$ Laplace tr. $\to$ non-hom. ODE $\to$hom. ODE) we can express the perturbation Φ as an expansion over the **quasinormal modes (QNMs)** frequencies ω_n

$$\Phi(t,y) = \sum_n e^{i\omega_n t} \mathrm{Res}\left(\frac{1}{W(s)}\right)\Bigg|_{\omega_n} \int_{-\infty}^{\infty} \psi_-(\omega_n, y_<)\psi_+(\omega_n, y_>)\mathcal{I}(\omega_n, y')\, dy'\,. \tag{12}$$

- QNMs ω_n are so found to be **zeros of wronskian of the fundamental regular solutions** at $y \to \pm\infty$ of the ODE [Nollert:1999], which also defines Q function as (3)

$$W[\psi_+, \psi_-](\omega_n) = 0\,, \qquad \text{with} \quad \psi_\pm(y) \to 0 \quad y \to \pm\infty\,. \tag{13}$$

Computing QNMs from Thermodynamic Bethe Ansatz

- The Q (or $Y = Q^2$) functions satisfy functional equations which can be inverted into the **Thermodynamic Bethe Ansatz (TBA)** for $\varepsilon(\theta) = -2\ln Q(\theta)$ (here for $SU(2)$ $N_f = 0$):

$$\varepsilon(\theta) = \frac{16\sqrt{\pi^3}}{\Gamma(\frac{1}{4})^2} e^{\theta} - 2\int_{-\infty}^{\infty} \frac{\ln\left[1+\exp\{-\varepsilon(\theta')\}\right]}{\cosh(\theta-\theta')} \frac{d\theta'}{2\pi}, \tag{14}$$

with $\varepsilon(\theta, P) \simeq 8P\theta$, $P > 0$ as $\theta \to -\infty$.

- The **QNMs condition** in gravity variables **reads also as**

$$\varepsilon(\theta_n - i\pi/2) = -i\pi(2n+1)\,. \tag{15}$$

- Through the TBA and quantization condition we have found a **new way to numerically compute QNMs**.

n	l	TBA	Leaver
0	0	$1.36912 - 0.504048i$	$1.36972 - 0.504311i$
0	1	$2.09118 - 0.501788i$	$2.09176 - 0.501811i$
0	2	$2.8057 - 0.501009i$	$2.80629 - 0.501000i$
0	3	$3.51723 - 0.500649i$	$3.51783 - 0.500634i$
0	4	$4.22728 - 0.500453i$	$4.22790 - 0.500438i$

Table: Comparison of QNMs of the D3 brane from TBA (14) (through (15) with $n = 0$), Leaver method (with $L = 1$).

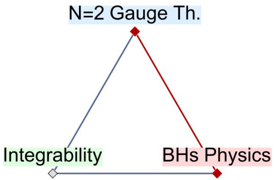
On the new $\mathcal{N} = 2$ gauge / gravity duality

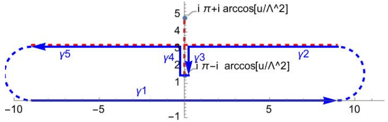
- Since proved a **relation between the Q function and the gauge periods**

$$Q(\theta, P) = \exp \frac{2\pi i}{\hbar} a_D(\hbar, u, \Lambda_0) \iff \varepsilon = -2 \ln Q = -\frac{4\pi i}{\hbar} a_D \,, \quad (16)$$

from our formalism **it follows immediately that QNMs are given by quantization conditions on the gauge periods**

$$\frac{1}{\hbar} a_D(i\hbar, -u, \Lambda_0) = \frac{i}{2}\left(n + \frac{1}{2}\right) . \quad (17)$$

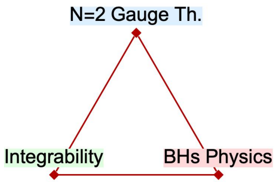




- (17) was previously found heuristically in [Aminov,Grassi,Hatsuda:2020] and allowed:
  - a **novel analytic characterization of QNMs**;
  - to **find new results for the BHs theory** (by many other authors);
  - an **unexpected application of Supersymmetry**.

Other applications of integrability

- Through ODE/IM we can define also **Baxter's T, $\tilde{T}$ functions** as wronskians of solutions at $y \to \pm\infty$ and derive TQ and T periodicity functional relations. From this:
  - since we have proven a relation between T and a gauge period, we give also a proof of the **alternative quantization condition for QNMs on** a for the $SU(2)$ $N_f = 0$ theory;
  - moreover we find that the **Couch-Torrence symmetry** corresponds to T **self-duality**.
- We notice that **much of the BH theory seems to go in parallel to the ODE/IM correspondence** construction and its 2D integrable field theory interpretation, beyond the determination of QNMs!
  - For instance, also the **greybody factor** that parametrizes the **Hawking radiation** seems to be **ratio of** Q**s**.



Details: Alternative QNMs quantizations condition for D3 brane

- From the relation between T and a it follows a proof of the **alternative quantization condition for quasinormal modes** (only for the $N_f = 0$ $SU(2)$ theory, that is the D3 brane)

$$a(\hbar, \Lambda_0) = n/2 \,, \qquad n \in \mathbb{Z} \,. \tag{18}$$

- Indeed, the TQ relation $T(\theta)Q(\theta) = Q(\theta - i\pi/2) + Q(\theta + i\pi/2)$ means $Q(\theta_n - i\pi/2) + Q(\theta_n + i\pi/2) = 0$. This and the QQ relation $Q(\theta + i\pi/2)Q(\theta - i\pi/2) = 1 + Q^2(\theta)$ actually fixes $Q(\theta_n + i\pi/2)Q(\theta_n - i\pi/2) = 1$ and then

$$Q(\theta_n \pm i\pi/2) = \pm i \tag{19}$$

are fixed, too. Again the QQ relation around θ_n forces $Q(\theta + i\pi/2) = i \pm Q(\theta) + \ldots$ and $Q(\theta - i\pi/2) = -i \pm Q(\theta) + \ldots$ up to smaller corrections (dots). Therefore, TQ relation imposes

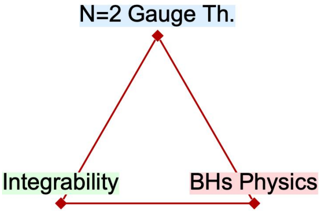
$$T(\theta_n) = 2\cos 2\pi a(\theta_n) = \pm 2 \,. \tag{20}$$

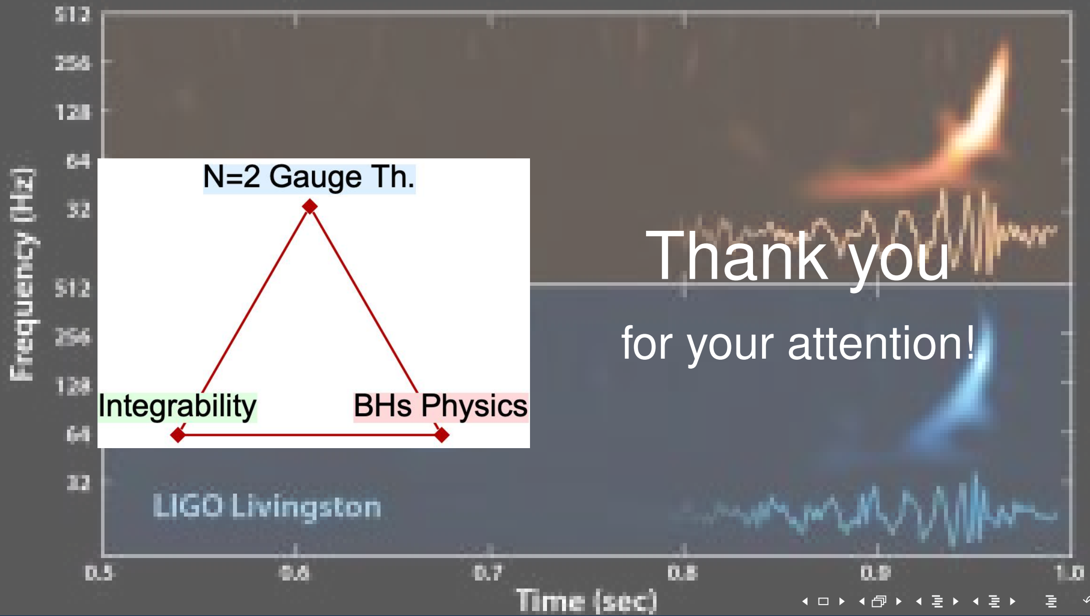
- We also have **another** relation and quantization with the **instanton gauge period** A_D

$$Q(\theta, P) = i\frac{\sin \frac{2\pi}{\hbar} A_D(\hbar, u, \Lambda_0)}{\sin \frac{2\pi}{\hbar} a} \quad , \quad Q(\theta_n) = 0 \quad \Longrightarrow \quad A_D(\hbar, u, \Lambda_0) = n/2 \,, \qquad n \in \mathbb{Z} \tag{21}$$

Conclusions and perspectives

- We have shown how **integrable models** when studied in **ODE/IM correspondence** approach can find
  - natural connection to (deformed) $\mathcal{N} = 2$ **SUSY gauge theory**
  - as well as to **black hole perturbation theory**
  - and shed light on the **relation** recently found **between the two**.
- This **new triality** besides being interesting in itself allows to:
  - **find new results** an all three sides of the correspondence
  - at the **non-perturbative exact level**.
- Especially for QNMs of BHs we give
  - **new analytic characterizations** from Q, Y, T functions
  - **new method of computation** as Thermodyn. Bethe Ansatz.
- $\mathcal{N} = 2$ gauge theories and black holes constitute **novel fields of application of integrability** and in these directions much extension work in either **model generalization** and **depth** should be done as well as **connection to other fields** (especially **holography**).





Thank you

for your attention!