Integrability as a new method for exact results on quasinormal modes of black holes

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From gauge to gravity & back

- Recently, a surprising connection between (Nekrasovi-Shatashvili (NS) deformed/quantum) $\mathcal{N}=2$ supersymmetric (SUSY) $SU(2)$ gauge theories and black holes (BHs) perturbation theory emerged.
- It was first found that quantization conditions on the gauge periods a, A_D (building blocks of prepotential in quantum Seiberg-Witten theory)

$$A_D(a,\hbar,\Lambda_n)=i\pi n \qquad n\in\mathbb{Z}\,, \tag{1}$$

  allow to compute the quasinormal modes (QNMs) frequencies $\omega_n\sim\Lambda_n$ of BHs (of the gravitational waves signal in the ringdown phase). [arXiv:2006.06111]
- This result is important because:
  1. constitutes a new analytic characterization of QNMs;
  2. in the growing subsequent literature it allowed to derive many new results for the BHs theory;
  3. constitutes a concrete, fruitful application of SUSY, beyond elementary particle physics.

Integrability & N=2 gauge th.

- Using the ODE/IM correspondence (see below) we connected basic integrability Q, T and Y functions to the quantum gauge theory periods a, a_D. For instance we proved the basic identifications: [1]

$$\begin{aligned} Q(\theta,P)&=\exp\{2\pi i a_D(\hbar,u,\Lambda)/\hbar\}\\ T(\theta,P)&=2\cos\{2\pi a(\hbar,u,\Lambda)/\hbar\}\,. \end{aligned} \tag{2}$$

  under the parameters correspondence

$$\frac{\hbar}{\Lambda}=\frac{\epsilon_1}{\Lambda}=e^{-\theta} \qquad \frac{u}{\Lambda^2}=\frac{1}{2}P^2e^{-2\theta}\,, \tag{3}$$

  here for the self-dual Liouville Integrable Model (IM) and $SU(2)$ $N_f=0$ gauge theory.
- This allowed also to find several new results on both sides, for instance:
  1. an exact non linear integral equation (Thermodynamic Bethe Ansatz, TBA) for the gauge periods;
  2. an interpretation of the integrability functional relations as new exact R-symmetry relations for the periods;
  3. new formulas for the local integrals of motion in terms of gauge periods.

  For instance, the TQ relation

$$T(\theta,u)=\frac{Q(\theta-i\pi/2,-u)+Q(\theta+i\pi/2,-u)}{Q(\theta,u)} \tag{4}$$

  turns out to be a new exact R-symmetry relation for the periods, reducing to the known asymptotic ones in the limit $\theta\to\infty$ (for the gauge periods expansion modes $a^{(n)}, a_D^{(n)}$)

$$a_D^{(n)}(-u)=i(-1)^n\left[-\mathrm{sgn}(\Im u)\,a_D^{(n)}(u)+a^{(n)}(u)\right]. \tag{5}$$

- This is a new kind of gauge/integrability correspondence and we developed it until now for the $SU(2)$ $N_f=0,1,2$ [1, 4] and $SU(3)$ $N_f=0$ $\mathcal{N}=2$ [2] (NS deformed) gauge theories, in correspondence with Liouville, Pertubed Hairpin, A_2 Toda IMs.

Integrability & quasinormal modes of black holes

- Perturbations Φ of a BHs are mathematically solutions of some PDE, which expands in quasi-normal modes (QNMs) ω_n as

$$\Phi(t,x)=\sum_n e^{i\omega_n t}\mathrm{Res}\left(\frac{1}{W(\omega)}\right)\Bigg|_{\omega_n}\int_{-\infty}^{\infty}\psi_-(\omega_n,x_<)\psi_+(\omega_n,x_>)\mathcal{I}(\omega_n,x')\,dx'\,, \tag{6}$$

  where x is the tortoise coordinate, $\mathcal{I}$ is given by initial data on the perturbation and $\psi_\pm$ are solutions of the ODE associated to the PDE through Laplace transform.
- Crucially, we noticed that QNMs are given by the zeros of the same wronksian which in ODE/IM correspondence (see below) defines the Baxter's Q function of some IM [3]

$$W(\omega_n)=W[\psi_+,\psi_-](\omega_n)=Q(\omega_n)=0\,. \tag{7}$$

  In particular, ODE (14) for the self-dual Liouville IM describes the scalar perturbation of the D3 brane

$$ds^2=H(r)^{-1/2}(-dt^2+d\mathbf{x}^2)+H(r)^{1/2}(dr^2+r^2d\Omega_5^2)\,, \qquad H(r)=1+L^4/r^4\,, \tag{8}$$

  under the change of parameters

$$r=Le^{\frac{y}{2}} \qquad \omega L=-2ie^{\theta} \qquad P=\frac{1}{2}(l+2)\,, \tag{9}$$

- Thanks to our identifications (2) we prove that QNMs are given by quantization conditions on gauge period a_D

$$2\pi i a_D(i\hbar,-u,\Lambda_n)=\pi\left(n+\frac{1}{2}\right)\,, \qquad n\in\mathbb{N} \tag{10}$$

  and prove the result (1), using also consistent relations between gauge periods a_D, A_D.

$$Q(\theta,P)=\exp\{2\pi i a_D(\hbar,u,\Lambda)/\hbar\}=i\frac{\sinh A_D(\hbar,a,\Lambda)/\hbar}{\sinh 2\pi i a/\hbar}\,, \tag{11}$$

- It follows also a new simple and effective method to numerically compute the quasinormal modes: the Thermodynamic Bethe Ansatz (TBA) for $\varepsilon=-2\ln Q$

$$\varepsilon(\theta)=\frac{16\sqrt{\pi^3}}{\Gamma(\frac{1}{4})^2}e^{\theta}-2\int_{-\infty}^{\infty}\frac{\ln[1+\exp\{-\varepsilon(\theta')\}]}{\cosh(\theta-\theta')}\frac{d\theta'}{2\pi}\,, \tag{12}$$

  which we compare with other standard methods and sometimes find convenient.

n	l	TBA	Leaver	WKB approx.
0	0	$1.36912-0.504048i$	$1.36972-0.504311i$	$1.41421-0.5i$
0	1	$2.09118-0.501788i$	$2.09176-0.501811i$	$2.12132-0.5i$
0	2	$2.8057-0.501009i$	$2.80629-0.501000i$	$2.82843-0.5i$
0	3	$3.51723-0.500649i$	$3.51783-0.500634i$	$3.53553-0.5i$
0	4	$4.22728-0.500453i$	$4.22790-0.500438i$	$4.24264-0.5i$

- The spacetimes for which we show this are in the simplest $SU(2)$ $N_f=0$ case the D3 brane in the $N_f=1,2$ case a generalization of extremal Reissner-Nordström (charged) BHs [3, 4].
- Anyway, the ODE/IM construction can be realized still for the $SU(2)$ $N_f=3,4$ theories, which correspond to non-extremal, asymptotically flat and asymptotically AdS BHs.

ODE/IM correspondence

- In this classic approach to integrability, the Q function of some Integrable Model (IM) is defined from the regular solutions at different singular points

$$Q=W[\psi_+,\psi_-] \qquad \psi_\pm(y)\to 0 \quad y\to\pm\infty \tag{13}$$

  of some ODE, like (self-dual Liouville IM) [1]

$$-\frac{d^2}{dy^2}\psi+\left[e^{2\theta}(e^y+e^{-y})+P^2\right]\psi=0\,. \tag{14}$$

- One can derive also T, Y functions as well as their functional and integral equations they satisfy.
- It allows to apply integrability to other different physical theories which share the ODE.

Possible future developments

- Besides models extension, it is very intriguing to study the applications of integrability beyond QNMs (*e.g.* the *greybody factor* seems a ratio of Qs.).
- Studying AdS BHs ($SU(2)$ $N_f=4$ gauge theory) would allow to extend the triple Integrability/Gauge/Gravity correspondence to a 4-fold correspondence through holography.

References

[1] D. Fioravanti, D. Gregori, arXiv:1908.08030

[2] D. Fioravanti, H. Poghosyan, R. Poghossian, arXiv:1909.11100

[3] D. Fioravanti, D. Gregori, arXiv:2112.11434

[4] D. Fioravanti, D. Gregori, H. Shu, to appear