Integrability and cycles of deformed $\mathcal{N} = 2$ gauge theory

Daniele Gregori

University of Bologna, NORDITA

INFN Strings Young Days
13th December 2021

Based on:
arXiv:1908.08030, arXiv:1909.11100 and to appear
with D. Fioravanti, H. & R. Poghossian and H. Shu.

Outline

Introduction to ODE/IM

- In the ODE/IM (Ordinary Differential Equations / Integrable Models) Correspondence, one can compute and derive all integrability functions (Baxter's Q, Y, T...) and functional relations starting from the solutions of some ODE related to some IM. [Dorey-Tateo:1999; Lukyanov-Bazhanov-Zamolodchikov:1999; Gaiotto-Moore-Neitzke:2010]
- It was discovered by Alexei Zamolodchikov [Zamolodchikov:2012] that the following ODE (Generalized Mathieu Equation, GME)

$$\left\{-\frac{d^2}{dy^2} + e^{2\theta}(e^{y/b} + e^{-yb}) + P^2\right\}\psi(y) = 0\,, \tag{1}$$

is associated to the Liouville integrable model (for L. coupling b, momentum P and rapidity θ) [Zamolodchikov:2006].

Introduction to deformed Seiberg-Witten theory

- In 4D $\mathcal{N} = 2$ SYM theories, to include instanton contributions, it is necessary to deform spacetime through two parameters, ϵ_1 and ϵ_2 (Ω-*background*). [Nekrasov:2004, Nekrasov-Okounkov:2006, Nekrasov:2009]
  - When both $\epsilon_1\,,\epsilon_2 \to 0$ we have the Seiberg-Witten (SW) original theory.
  - An intermediate limit which we will study is the Nekrasov-Shatashvili (NS): $\epsilon_1 = \hbar \neq 0$, $\epsilon_2 \to 0$ [Nekrasov-Shatashvili:2009].
- The NS deformed/quantized SW differential is obtained from the solution of the Mathieu equation

$$-\frac{\hbar^2}{2}\frac{d^2}{dz^2}\psi(z) + [\Lambda^2 \cos z - u]\psi(z) = 0\,. \tag{2}$$

where u parametrizes the moduli space of vacua and Λ is a scaling parameter. [Alday-Gaiotto-Tachikawa:2010; Gaiotto:2013; Awata-Yamada:2010]

- The deformed prepotential $\mathcal{F}_{\mathrm{NS}}$ (logarithm of the partition function) may be derived by eliminating u between the two deformed cycles (periods)

$$a(\hbar, u, \Lambda) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{P}(z; \hbar, u, \Lambda)\, dz\,, \quad a_D(\hbar, u, \Lambda) = \frac{1}{2\pi} \int_{-\arccos(u/\Lambda^2) - i0}^{\arccos(u/\Lambda^2) - i0} \mathcal{P}(z; \hbar, u, \Lambda)\, dz \tag{3}$$

integrals of the quantum SW differential $\mathcal{P}(z) = -i \frac{d}{dz} \ln \psi(z)$.

- Usually, in the literature only their $\hbar \to 0$ asymptotic expansions (with coeff. $a^{(n)}$, $a_D^{(n)}$) are studied (obtained by WKB approximation on ODE). [Mironov-Morozov:2010]

Integrability-Gauge fundamental correspondence

- The self-dual $(b = 1)$ GME is known in literature as modified Mathieu equation:

$$\left\{-\frac{d^2}{dy^2} + 2e^{2\theta}\cosh y + P^2\right\}\psi(y) = 0\ , \tag{4}$$

- This equation can be related to the Mathieu equation which quantizes the SW differential in the NS limit (2) by the change of var. $z = -iy - \pi$ and the parameters correspondence

$$\frac{\hbar}{\Lambda} = \frac{\epsilon_1}{\Lambda} = e^{-\theta} \qquad \frac{u}{\Lambda^2} = \frac{1}{2}\frac{P^2}{e^{2\theta}}\ . \tag{5}$$

- We (D. Fioravanti and D. Gregori - arXiv:1908.08030) have found that the relations:

$$Q(\theta, P^2) = \exp 2\pi i a_D(\epsilon_1, u)\ , \tag{6}$$

$$T(\theta, P^2) = 2\cos 2\pi a(\epsilon_1, u)\ . \tag{7}$$

between the two NS deformed Seiberg-Witten cycle periods a, a_D for pure $(N_f = 0)$ SU(2) $\mathcal{N} = 2$ SYM and the Baxter's Q and T functions of the self-dual Liouville integrable model.

Proof of the fundamental identification for Q

- We have proven relation (6) analytically by studying the properties of the solution $\mathcal{P}(y) = -i\frac{d}{dy}\ln\psi(y)$ of the Riccati equation, equivalent to the ODE (4) with boundary condition given by the (double) limit $y \to +\infty$ of the SW leading $\hbar \to 0$ order. [Fioravanti,Gregori:2019]

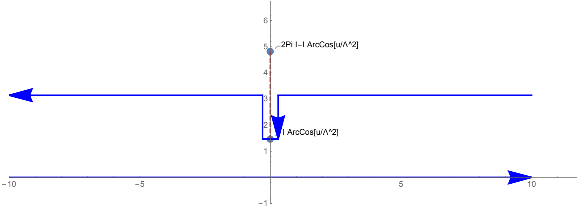


Figure: Integration contour in the y complex plane for the proof.

$$\underbrace{\int_{-\infty}^{+\infty} \mathcal{P}_{reg}(y)\, dy}_{\ln Q(\theta, P^2)}\,, = \underbrace{\int_{-\arccos(u/\Lambda^2)-i0}^{+\arccos(u/\Lambda^2)-i0} \mathcal{P}(z)\, dz}_{2\pi i a_D(\hbar, u)} \tag{8}$$

$$Q(\theta, P^2) = \exp 2\pi i a_D(\epsilon_1, u)\,, \tag{9}$$

Proof of the fundamental identification for T

- Alexei Zamolodchikov proved numerically a relation between the $b = 1$ T and the Floquet index ν of the Mathieu equation, such that $\psi(z + 2\pi) = e^{2\pi\nu}\psi(z)$. [Zamolodchikov:2012]

$$T = 2\cosh 2\pi\nu\,. \tag{10}$$

- He did not recognise that $\nu = ia$ (NS limit of $\mathcal{N} = 2$ SYM not yet existing), but we did it, thus establishing another connection to gauge theory [Fioravanti,Gregori:2019]

$$T(\theta, P^2) = 2\cos 2\pi a(\epsilon_1, u)\,. \tag{11}$$

Gauge TBA and R-symmetry

- The self-dual Liouville QQ relation for $Q(\theta, u) \equiv Q(\theta, P^2)$ translates in the gauge variables as

$$1 + Q^2(\theta, u) = Q(\theta - i\pi/2, -u)Q(\theta + i\pi/2, -u) \tag{12}$$

and can be inverted to obtain a TBA for the dual period $(e^{-\varepsilon(\theta,u)} = Q^2(\theta, u))$

$$\varepsilon(\theta, u) = -4\pi i a_D^{(0)}(u)\frac{e^\theta}{\Lambda} - 2\int_{-\infty}^{\infty} \frac{\ln\left[1 + \exp\{-\varepsilon(\theta', -u)\}\right]}{\cosh(\theta - \theta')}\frac{d\theta'}{2\pi} \tag{13}$$

- The TBA enjoys the $\mathbb{Z}_2$ R-symmetry of the moduli space $u \leftrightarrow -u$, which is nothing but the discrete symmetry used for the ODE/IM construction.

- The TBA for the Liouville integrable model (for $b = 1$) is

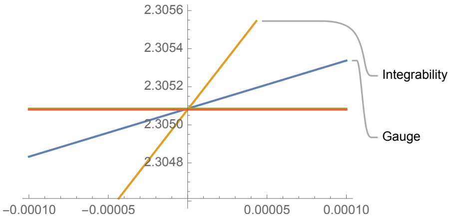
$$\varepsilon(\theta) = \frac{16\sqrt{\pi^3}}{\Gamma(\frac{1}{4})^2} e^{\theta} - 2\int_{-\infty}^{\infty} \left[\frac{1}{\cosh(\theta-\theta')}\right] \ln\left[1+\exp\{-\varepsilon(\theta')\}\right] \frac{d\theta'}{2\pi}, \tag{14}$$

with boundary condition $\varepsilon(\theta, P^2) \simeq +8P\theta$, $P > 0$, at $\theta \to -\infty$.

- The solution $\varepsilon(\theta, u)$ of the gauge TBA (13) and that $\varepsilon(\theta, P^2)$ of the integrability TBA (14) match

$$\varepsilon(\theta_0, u) = \varepsilon(\theta_0, P^2) \text{ when } P^2 = 2ue^{2\theta_0}/\Lambda^2, \tag{15}$$

that is, we verified numerically the relation (6).



Non perturbative $\mathbb{Z}_2$ symmetry relations

- The self-dual Liouville TQ relation and T periodicity relation read in gauge variables

$$T(\theta, u) = \frac{Q(\theta - i\pi/2, -u) + Q(\theta + i\pi/2, -u)}{Q(\theta, u)}\,, \tag{16}$$

$$T(\theta, u) = T(\theta - i\pi/2, -u)\,. \tag{17}$$

- These relations appear to be a non-perturbative exact generalization of the perturbative $(\theta \to +\infty, \hbar \to 0)$ $\mathbb{Z}_2$ R-symmetry relations for the periods [Bilal-Ferrari:1996, Basar-Dunne:2015]

$$a_D^{(n)}(-u) = i(-1)^n \left[-\mathrm{sgn}\,(\mathrm{Im}\, u)\, a_D^{(n)}(u) + a^{(n)}(u) \right], \tag{18}$$

$$a^{(n)}(-u) = -i(-1)^n \mathrm{sgn}\,(\mathrm{Im}\, u)\, a^{(n)}(u)\,. \tag{19}$$

Perturbative periods and local integrals of motion

- By taking a double limit $\theta, P^2 \to +\infty$ on the integrability asymptotic expansion of Q, the LIMs I_{2n-1} resum to the perturbative periods $a_D^{(n)}$.

$$2\pi i a_D^{(n)}(u,\Lambda) = -\Lambda^{1-2n}\sum_{k=0}^{\infty} 2^k C_{n+k}\Upsilon_{n+k,k}\left(\frac{u}{\Lambda^2}\right)^k . \tag{20}$$

  where $I_{2n-1}(b=1,P^2) = \sum_{k=0}^{n}\Upsilon_{n,k}P^{2k}$.
- We have derived from the series (20) the quantum Picard-Fuchs equations at all perturbative orders (through an algorithm) for $a^{(n)}$ and $a_D^{(n)}$. For instance

$$\left\{(u^2-\Lambda^4)\frac{\partial^2}{\partial u^2} + 6u\frac{\frac{u^2}{\Lambda^4}+\frac{111}{8}}{\frac{u^2}{\Lambda^4}+\frac{325}{32}}\frac{\partial}{\partial u} + \frac{21}{4}\frac{\frac{u^2}{\Lambda^4}+\frac{689}{32}}{\frac{u^2}{\Lambda^4}+\frac{325}{32}}\right\} a_D^{(2)}(u,\Lambda) = 0 . \tag{21}$$

- Viceversa, we can interpret in integrability this equations as fixing the LIMs for $b=1$.

Generalisations

- We have described our gauge-integrability connection just for the simplest $SU(2)$ with $N_f = 0$ case, but we have evidence that it is of much more general validity.
  - For the case of $SU(3)$ with $N_f = 0$, D.Fioravanti, R.Poghossian and H.Poghosyan have found a relation between the 3 periods and the T function of the A_2 Toda Integrable model, which is a generalisation of (7). [Fioravanti-Poghossian-Poghosyan:2019]
  - For the case of $SU(2)$ with $N_f = 1$, D. Fioravanti, D.Gregori and H. Shu have found a relation between the 2 periods and the Y of the Integrable Perturbed Hairpin model. [Fioravanti-Gregori-Shu:to appear]
- A different kind of connection between Q and Y functions and gauge periods has been found in [Grassi-Gu-Marino:2019, Grassi-Hao-Neitzke:2021]. However, it turns out that they are different, since they involve different periods, generated by the instanton prepotential rather than by the cycle integrals.

Summary and dictionary

- In conclusion, the powerful ODE/IM correspondence has been revealing a very suggestive connection between the 2D quantum integrable models and 4D NS (ϵ_1)-deformed Seiberg-Witten $\mathcal{N} = 2$ supersymmetric gauge theories.
- The ODE/IM correspondence yields a natural quantisation scheme for general SW theory: the SW differential becomes quantised as differential operator whose cycles (or monodromies) are encoded into the connection coefficients.

2D Integrability	4D NS-def. $\mathcal{N} = 2$ SYM
Q, Y function	(dual) period a_D
T function	period a
ODE/IM discrete sym.	$\mathbb{Z}_n$ R-symmetry
Q, Y system	TBA for gauge period
TQ relation	R-symmetry exact rel. for a_D
T periodicity	R-symmetry exact rel. for a
L.I.M. I_{2n-1}	quantum P.F. eq. for $a^{(n)}, a_D^{(n)}$
n-pert. reg. TBA	resum. instanton contr.

Conclusions and perspectives

- It would be also interesting to explore:
  - the correspondence with higher $SU(2)$ flavours $N_f = 2, 3, 4$
  - the application of these results in the study of quasi-normal-modes frequencies of black holes merging; [Aminov-Grassi-Hatsuda:2020, Bianchi-Consoli-Grillo-Morales:2021]
  - the interpretation of Liouville integrable structure within AGT duality; [Alday-Gaiotto-Tachikawa:2010, Bonelli,Iossa,Lichtig,Tanzini:2021]
  - the problem of wall-crossing of TBA, that is, the gauge TBA also in the weak coupling regime (should be related to TQ relation);
  - the generalization to the full Omega background, with both $\epsilon_1, \epsilon_2 \neq 0$ (a PDE/IM correspondence would need to be developed);
  - ...

THANK YOU!