From Integrability to Supersymmetry and Black Holes

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Based on: arXiv:25**.*****, arXiv:2208.14031, arXiv:2112.11434, arXiv:1908.08030

with D. Fioravanti, R. Mahanta, M. Rossi, H. Shu

Introduction

Evolution of theoretical physics

How to interpret the “phase transition” in arXiv [hep-th] title words?

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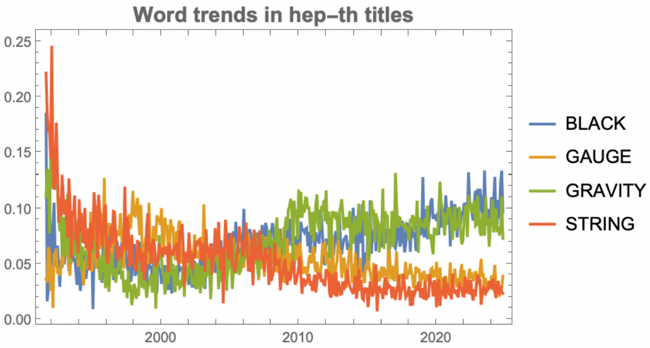

To draw such plots* just use my **ArXivExplore paclet**!

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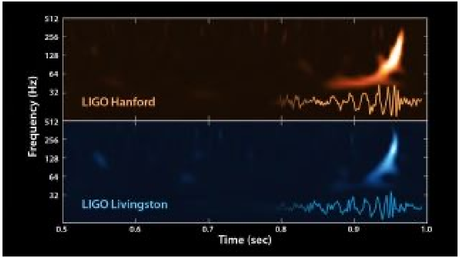
*as well as for computing **all-ArXiv** title/abstract **word statistics**; TeX source, **formulae**, **citations** extraction; **Neural Networks** for classification/recommendation/clustering; **LLM** concept **explanations** & **author reports**.

GWs: “A new window to the universe”

In the last few years, **gravitational waves (GWs)** detections have opened the way for **gravitational phenomenology**.

These new observations promise to revolutionise even **fundamental physics**, allowing the search of **deviations from general relativity (GR)** in previously unexplored regimes.



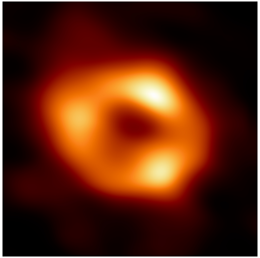


Figure: LIGO, Event Horizon

Since **BHs are the simplest and most perfect objects** in gravity, one may ask

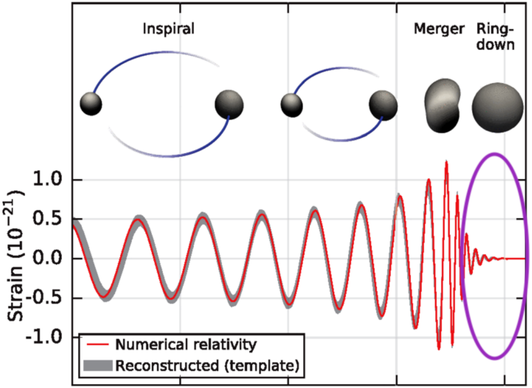
Could BHs hold the same surprises that the electron and the hydrogen atom did when they started to be experimentally probed? *[Cardoso-Pani, arXiv:1709.01525]*

Colliding BHs and quasinormal modes

A **black hole collision** can be divided in 3 phases: **inspiral, merger** and **ringdown**.

QNMs can be observed as the complex frequencies of the **damped oscillations of spacetime** in the final - **ringdown** - phase of black holes (BHs) merging.

[Nollert, Class.Quant.Grav. 16 (1999); Berti-Cardoso-Starinets, arXiv:0905.2975]



Black holes perturbations can be expressed in terms of solutions of **ODEs shared with other physics fields!**

The need of new methods

- Even if QNMs for GR BHs are already known, it is still an open problem to compute them in **modified gravity**. Specifically, the common methods are:
  - **Continued fraction** method: exact but laborious;
  - **WKB approximation**: easy but rough, asymptotic, to be resummed;
  - **Numeric simulations**: case by case, sometimes unstable, with little insight.

Therefore developing **new analytic characterizations and exact numerical methods** for QNMs is very important.

- QNMs also present **theoretical problems even in GR** like their instability under small perturbations of the background.

$$\frac{1}{4x^2(x-1)^3}\sum_{n=0}^{2}\sum_{m=0}^{2-n} A_{n,m}\, g^{(n)}(x)\,(i\,\Omega)^m = 0\,. \qquad (21)$$

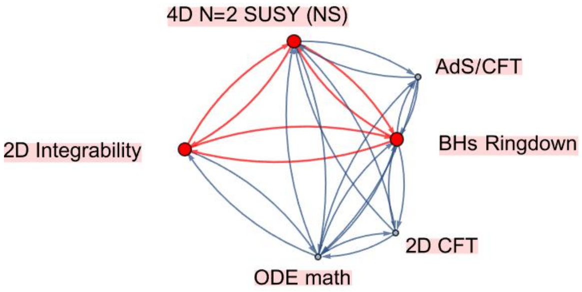
Here $g^{(n)}(x)$ is the n-th derivative of the function $g(x)$. The real coefficients $A_{n,m}(x)$ are

$$\begin{aligned}
A_{0,0} =& -x^2(x-1)^2\,[4(\ell-1)(\ell+2)\mathcal{M}_0\,\mathcal{S}_0^3\\
&-x(8\mathcal{N}_0\,\mathcal{S}_0' + x\mathcal{N}_0'\,\mathcal{S}_0' + 2x\mathcal{N}_0\,\mathcal{S}_0'')\,\mathcal{M}_0\,\mathcal{W}_0\,\mathcal{S}_0\\
&+x(2\mathcal{S}_0 - x\mathcal{S}_0')\,\mathcal{M}_0'\,\mathcal{N}_0\,\mathcal{W}_0\,\mathcal{S}_0\\
&+3x^2\mathcal{M}_0\,\mathcal{N}_0\,\mathcal{W}_0\,\mathcal{S}_0'^2\\
&+2(4\mathcal{N}_0 + x\mathcal{N}_0')\,\mathcal{M}_0\,\mathcal{W}_0\,\mathcal{S}_0^2]\,\mathcal{P}_0\\
A_{0,1} =& -2x^2\,[2(\delta + (x^2-2x)(\gamma+\delta))\mathcal{M}_0\,\mathcal{N}_0\\
&+(x-1)B(x)(\mathcal{M}_0\,\mathcal{N}_0)']\,\mathcal{P}_0\,\mathcal{W}_0\,\mathcal{S}_0^2\,,\\
A_{0,2} =& +4\,[B^2(x)\,\mathcal{M}_0\,\mathcal{N}_0\,\mathcal{P}_0 - (x-1)^2\,\mathcal{S}_0]\,\mathcal{W}_0\,\mathcal{S}_0^2\,,\\
A_{1,0} =& +2x^3(x-1)^2\,[x(\mathcal{M}_0\,\mathcal{N}_0)' + 4\mathcal{M}_0\mathcal{N}_0]\,\mathcal{P}_0\,\mathcal{W}_0\,\mathcal{S}_0^2\,,\\
A_{1,1} =& -8x^2(x-1)\,B(x)\,\mathcal{M}_0\,\mathcal{N}_0\,\mathcal{P}_0\,\mathcal{W}_0\,\mathcal{S}_0^2\,,\\
A_{2,0} =& +4x^4(x-1)^2\,\mathcal{M}_0\,\mathcal{N}_0\,\mathcal{P}_0\,\mathcal{W}_0\,\mathcal{S}_0^2\,,
\end{aligned}$$

Figure: ODE for axial perturbation in Einstein scalar Gauss-Bonnet theory [arXiv:2404.11583]

Plan of the talk

1. Introduction
2. SW theory and BHs
3. ODE/IM correspondence and BHs
   - Introduction to ODE/IM correspondence
   - QNMs as Bethe roots
   - Thermodynamics Bethe Ansatz
   - Solving quantum Wronskians of gauge theories
4. Conclusions

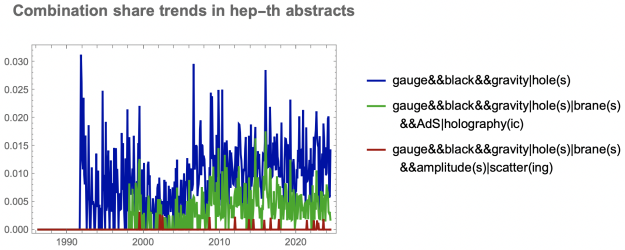


SW theory and BHs

To the rescue of supersymmetry?

We see an increasing of article abstracts talking simultaneously of **gauge theory and black holes**.

Actually, it is mostly due to **AdS/CFT** and NOT to the well known community of **scattering amplitudes**, modeling the inspiral phase of black holes as particle collision.

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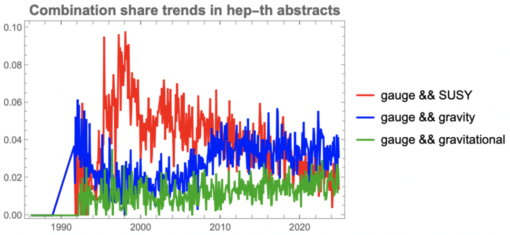
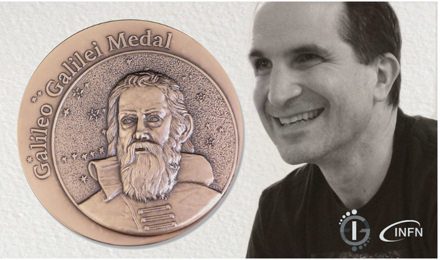


An even smaller community works differently with **Seiberg-Witten theory** on (mostly) the ringdown phase...

Are AdS black holes more **physical** than supersymmetric particles?

Galileo and Maldacena

At least twice in my short scientific life - during formal meetings at Bologna University - **I have been publicly attacked for proposing the study of ($\mathcal{N} = 2$) supersymmetry**...



Similarly - according to rumours - someone complained for the Galileo Galilei award to Prof. Maldacena, on the basis of **"not being an experimental physicist, like Galileo was"**...

...but in the *Saggiatore* we read also "the book of nature is written in..."

"The book of nature is written in mathematical language!"

Generally in both **General Relativity** and **modified gravity**, BHs perturbations obey different kind of ODEs.

- Perturbations of **asymptotically AdS_5 BHs** obey **Heun** equation

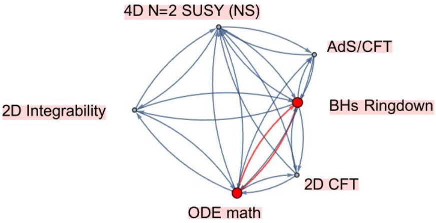
$$w''(z)+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\alpha+\beta-\gamma-\delta+1}{z-a}\right)w'(z)+\frac{\alpha\beta z-q}{z(z-1)(z-a)}w(z)=0\,. \qquad (1)$$

- for 4D **asymptotically flat BHs**, **Confluent Heun** equation (CHE)

$$w''(z)+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\epsilon\right)w'(z)+\frac{\alpha z-q}{z(z-1)}w(z)=0\,. \qquad (2)$$

- for 4D **Extremal** flat BHs, **Doubly Confluent Heun** eq. (DCHE)

$$w''(z)+\left(\frac{\delta}{z^2}+\frac{\gamma}{z}+1\right)w'(z)+\frac{\alpha z-q}{z^2}w(z)=0\,. \qquad (3)$$



- Caveat: **rotating BHs** involve **coupled ODEs** (separable PDEs)!
- The inspiral phase is governed by the **non-homogenous** same ODEs!

$\mathcal{N} = 2$ SUSY primer

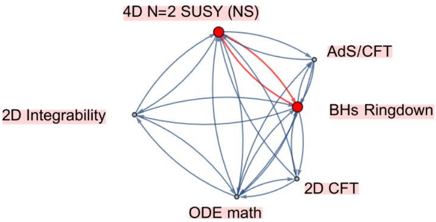
- **Seiberg-Witten (SW) theory** is the effective theory for $\mathcal{N} = 2$ SUSY gauge theory. To compute **instanton contributions**, it is convenient to deform spacetime in terms of two parameters $\epsilon_1, \epsilon_2 \in \mathbb{C}$ (**Ω-background**).
[Seiberg-Witten, hep-th/9407087; Nekrasov, hep-th/0206161]
- If $\epsilon_2 \to 0$ - **Nekrasov-Shatashvili (NS) limit** - the prepotential $\mathcal{F}$ can obtained from monodromies of ODEs of Schrödinger time-independent type with $\epsilon_1 = \hbar \neq 0$ (for $SU(2)$)

$$-\hbar^2 \frac{d^2}{dy^2}\psi(y) + \left[U(y; \mathbf{m}, \Lambda) - u\right]\psi(y) = 0\,, \tag{4}$$

[Nekrasov-Shatashvili, arXiv:0908.4052]

No jokes: Schwarshild black hole!

The quantum SW curve for $N_f = 3$ $SU(2)$ **gauge th.** can be mapped to the Regge-Wheeler equation for linear **perturbations of the Schwarschild BH!**



$$-\hbar^2 \frac{d^2}{dy^2}\psi(y) + \Big\{ e^{2y}\Lambda_3 \left(4(m_1 - m_2)^2\right) \tag{5}$$
$$+ \left(\Lambda_3^2 - 24\Lambda_3 m_3 + 64u\right) + 4e^y\sqrt{\Lambda_3}\left(-2\hbar^2 + 8m_1 m_2 + \Lambda_3 m_3 - 8u\right)$$
$$+ 4e^{-y}\sqrt{\Lambda_3}\left(8m_3 - \Lambda_3\right) + 4\Lambda_3 e^{-2y}\Big\} \frac{1}{16\left(\sqrt{\Lambda_3}e^y - 2\right)^2}\psi(y) = 0$$

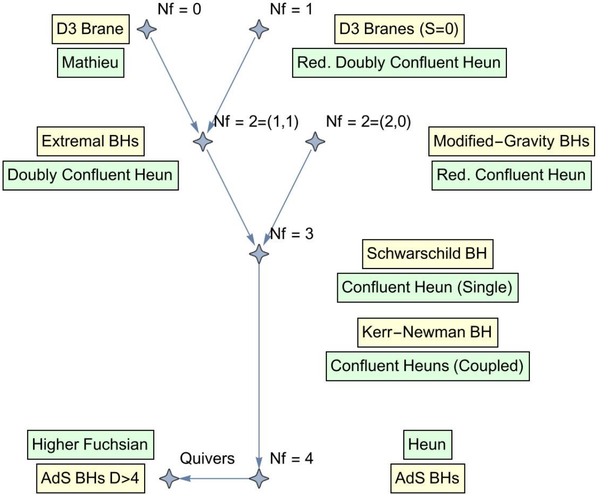
$$\Lambda_3 = -16iM\omega\,, \qquad u = l(l+1) - 8M^2\omega^2 + \frac{1}{4}\,, \qquad \Downarrow \qquad r = \frac{4M}{\Lambda_3}e^{-y}\,,$$

$$\hbar = 1\,, \qquad m_1 = s - 2iM\omega\,, \qquad m_2 = -s - 2iM\omega\,, \qquad m_3 = -2iM\omega\,,$$

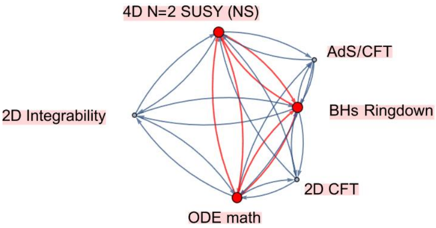
$$\left[f(r)\frac{d}{dr}f(r)\frac{d}{dr} + \omega^2 - f(r)\left[\frac{l(l+1)}{r^2} + (1-s^2)\frac{2M}{r^3}\right]\right]\phi(r) = 0\,, \tag{6}$$

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \quad \text{with } f(r) = 1 - 2M/r$$

General picture of new gauge - gravity correspondence



$SU(2)$ $\mathcal{N} = 2$ (NS) gauge theories, Heun ODEs and BHs models.



[Aminov-Grassi-Hatsuda, arXiv:2006.06111, Bianchi-Consoli-Grillo-Morales, arXiv:2109.09804]

Nekrasov-Shatashvili instanton series

- The NS prepotential can be obtained as a **series in the instanton coupling** Λ

$$\mathcal{F}(\Lambda, a, \hbar) = \mathcal{F}^{pert}(a, \hbar) + \sum_{n=1}^{\infty} \mathcal{F}_n^{inst}(a, \hbar)\Lambda^{(4-N_f)n} . \tag{7}$$

- The **gauge period** a is recovered from the **ODE moduli parameter** u through

$$u = \Lambda \frac{\partial \mathcal{F}(\Lambda, a, \hbar)}{\partial \Lambda} . \tag{8}$$

- Also the **gauge dual instanton period** A_D is defined as

$$A_D = \frac{\partial \mathcal{F}(\Lambda, a, \hbar)}{\partial a} \tag{9}$$

and in the $\hbar \to 0$ SW limit, $A_D \simeq \frac{1}{\hbar} A_D^{(0)}$, with $A_D^{(0)} = a_D^{(0)} + 2\pi i a^{(0)}$.

Nice analytical expressions!

- For example for $SU(2)$ gauge theory with $N_f = 3$ matter flavours, at **second order** we get:

$$
\begin{aligned}
\mathcal{F}^{inst}(\Lambda_3, m_1, m_2, m_3, a, \hbar) = & -\frac{m_1 m_2 m_3}{4(4a^2-\hbar^2)}\Lambda_3 \\
& + \Bigg[\frac{-256a^8 - 256a^6\left(m_3^2+m_1^2+m_2^2-2\hbar^2\right) + 96a^4\left(2m_3^2\hbar^2 + 2m_2^2\left(4m_3^2+\hbar^2\right) + 2m_1^2\left(4m_3^2+4m_2^2+\hbar^2\right) - 3\hbar^4\right)}{8192\left(a^2-\hbar^2\right)\left(4a^2-\hbar^2\right)^3} \\
& + \frac{-16a^2\left(3m_3^2\hbar^4 + 3m_2^2\left(8m_3^2\hbar^2+\hbar^4\right) + m_1^2\left(8m_2^2\left(10m_3^2+3\hbar^2\right) + 3\left(8m_3^2\hbar^2+\hbar^4\right)\right) - 4\hbar^6\right)}{8192\left(a^2-\hbar^2\right)\left(4a^2-\hbar^2\right)^3} \\
& + \frac{4m_3^2\hbar^6 + 4m_2^2\hbar^6 + 48m_3^2m_2^2\hbar^4 + 4m_1^2\hbar^2\left(12m_3^2\hbar^2 - 4m_2^2\left(28m_3^2-3\hbar^2\right)+\hbar^4\right) - 5\hbar^8}{8192\left(a^2-\hbar^2\right)\left(4a^2-\hbar^2\right)^3}\Bigg]\Lambda_3^2 + O(\Lambda_3^3)\,. \qquad (10)
\end{aligned}
$$

- The computation algorithm is based on **combinatorics on Young diagrams**

Nice **analytical** expressions...but the terms are infinite and the computation time for higher orders **grows exponentially**!

Rescued by supersymmetry?

It was found heuristically that **quasinormal modes (QNMs) are given by quantization conditions on the gauge periods**

$$A_D(\Lambda_n, \mathbf{m}, u) = 2i\pi\hbar\, n \quad \text{for } n \in \mathbb{N} \tag{11}$$

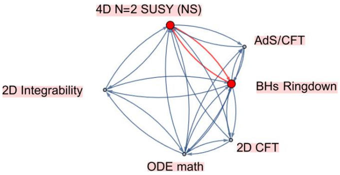
$$\Downarrow$$

$$\text{QNMs} \quad \omega_n \propto \Lambda_n$$

[Aminov-Grassi-Hatsuda, arXiv:2006.06111]

A small field is growing on this result, since:

- qSW theory turns out to be a **very effective method for QNMs computation**;
- it also provided **new theoretical results**.

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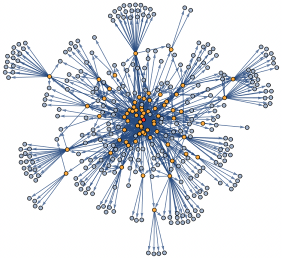


ODE/IM correspondence and BHs

On quantum integrability

- The hallmark of quantum integrability is the presence of **infinite integrals of motion commuting** with each other

$$[\mathbf{I}_n, \mathbf{I}_m] = 0 \,. \tag{12}$$

- The IM I_{2n-1} are also asymptotic expansion coefficients, for large spectral parameter $\lambda = e^{\theta}$, of the **Baxter's** Q, T **operators**

$$\ln \mathbf{Q}(\theta) \simeq -C_0 e^{\theta} - \sum_{n=1}^{\infty} e^{-n\theta} C_n \mathbf{I}_n \qquad \theta \to +\infty \,, \tag{13}$$

$$[\mathbf{Q}(\theta), \mathbf{T}(\theta)] = 0 \,, \tag{14}$$

[Bazhanov-Lukyanov-Zamolodchikov, hep-th/9412229, hep-th/9604044]

- Their eigenvalues, the Q, T **functions** satisfy **functional relations**, like **quantum wronskian** QQ or **Baxter's** TQ, which given suitable **asymptotic behaviours** have **unique solutions**.

ODE/IM correspondence essentials

- The first ODE/IM instance found was for the **anharmonic oscillator ODE**, with $\alpha \geq -1$

$$-\frac{d^2}{dx^2}\psi(x) + \left(x^{2\alpha} + \frac{l(l+1)}{x^2} - E\right)\psi(x) = 0\,. \tag{15}$$

- The **spectral determinants** are also computed as **Wronskians**

$$Q^+(E,l) = Q_0^+ \prod_{k=0}^{\infty}\left(1 - \frac{E}{E_{2k}}\right) = W[\psi_{+,0},\psi_{-,0}] \text{ with } \begin{cases} \psi_{+,0}(x) \to 0\,, & x \to \infty \\ \psi_{-,0}(x) \to 0\,, & x \to 0 \end{cases}. \tag{16}$$

- $Q^+(E,l)$ can be **computed** as **integral** of the solution $\mathcal{P}(x) = -i\frac{d}{dx}\ln\psi(x)$ of the **Riccati** ODE equivalent to (15)

$$Q^+(E,l) = \lim_{x\to 0}[x^l\psi_{+,0}(x)] = \exp\left\{i\int_0^\infty [\mathcal{P}(x) - \text{regulariz.}\,]\,dx\right\}\,. \tag{17}$$

- Through **discrete symmetries** of the ODE, we can define other solutions in $\mathbb{C}$, with $\omega = e^{\frac{i\pi}{\alpha+1}}$

$$\Omega_+ : x \to \omega^{-1} x \,, \quad E \to \omega^2 E \quad \Longrightarrow \quad \psi_{+,k} \equiv \Omega_+^k \psi_{+,0} = \psi_{+,0}(\omega^{-k} x, \omega^{2k} E, l) \,, \tag{18}$$

so we can write **lateral connection relations** in terms **Stokes multipliers** $T_{k/2}$

$$\psi_{-1}(x) = T_{k/2}(E,l)\psi_{k-1}(x) + \tilde{T}_{k/2}(E,l)\psi_k(x) \quad \Longrightarrow \quad T_{k/2}(E,l) = W[\psi_{-1}, \psi_k] \,. \tag{19}$$

- Similarly defining $\psi_{-,1}(x,E,l) \equiv \psi_{-,0}(x,E,-l-1)$ and $Q^-(E,l) = Q^+(E,-l-1)$ from **central connection relations** (between $\psi_{-,k}$ and $\psi_{+,k}$) we obtain QQ **system**

$$\omega^{1/2} Q^+(\omega^{-1}E) Q^-(\omega E) - \omega^{-1/2} Q^+(\omega E) Q^-(\omega^{-1}E) = 2i \,. \tag{20}$$

- Similarly also TQ and T systems.
- $Q^\pm$, $T_{k/2}$ realize **vacuum** Q, T **functions** of **Scaling 6-vertex IM!** [Dorey-Tateo, hep-th/9812211, Bazhanov-Lukyanov-Zamolodchikov, hep-th/9812247]

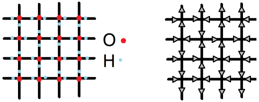
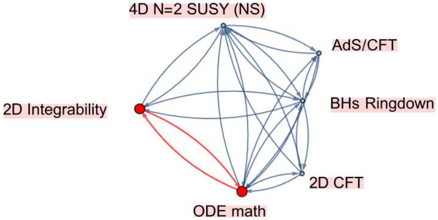


Figure: (Wikicommons/Moali)

The ODE/IM correspondence

- The **ODE/IM correspondence** is an approach to integrability, in which the Q, T functions and all the (QQ, TQ...) **integrable structures** of some Integrable Model (IM) can be **derived from an Ordinary Differental Equation** (ODE), even without knowing a priori the corresponding IM. [Dorey-Tateo, hep-th/9812211, Bazhanov-Lukyanov-Zamolodchikov, hep-th/9812247, Dorey-Dunning-Tateo, hep-th/0703066]

ODE	IM
Spectral determinant	Q
Stokes multipliers	T_k
Central Connect. rels.	QQ relation
Lateral Connect. rels.	TQ, T-syst
ODE parameters	IM parameters
ODE ind. var. x	... cf. below



ODE/IM allows to apply integrability to very different physical theories!

Other ODE/IM examples: Liouville

- We can reach the regime $\alpha < -1$, by a formal change of variable, leading to the ODE for the to **Liouville model** , that is CFT central charge $c = 1 + (b + 1/b)^2 \geq 1$

$$-\frac{d^2}{dy^2}\psi(y) + \left[e^{2\theta}(e^{y/b} + e^{-by}) + P^2\right]\psi(y) = 0\,, \tag{21}$$

which reduces to (modified) Mathieu equation for $b = 1$ ($c = 25$).
[Fioravanti-DG, arXiv:1908.08030]

- Liouville model is
  - the **2D quantum gravity** theory
  - dual to 4D $\mathcal{N} = 2$ SUSY through **AGT duality**

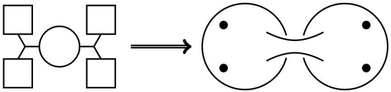


Figure: AGT duality quiver (NLab)

We make a different connection through ODE/IM!

Other ODE/IM examples: Perturbed Hairpin, Paperclip

- **Perturbed Hairpin IM** (for $n = 1, 2$) corresponds to Doubly Confluent Heun eq.

$$-\frac{d^2}{dx^2}\psi(x) + \left[2kqe^x + k^2(e^{2x} + e^{-nx}) + P^2\right]\psi(x) = 0\,. \tag{22}$$

- **Paperclip IM** (for $n = 2$) corresponds to the Confluent Heun eq.

$$-\frac{d^2}{dx^2}\psi(x) + \left[k^2\left(e^x + 1\right)^n - \frac{p^2 e^x}{e^x + 1} - \frac{\left(q^2 - \frac{1}{4}\right)e^x}{\left(e^x + 1\right)^2}\right]\psi(x) = 0\,. \tag{23}$$

- They are both related **2D IQFTs** with **non-conformal boundary interaction**.
[Fateev-Lukyanov, hep-th/0510271; Lukyanov-Vitchev-Zamolodchikov, hep-th/0312168]

Fig. 2. The paperclip formed by a junction of two hairpins.

On the origin of ODE/IM

- For a long time, ODE/IM has been regarded as either **mysterious correspondence** or just a mathematical coincidence.
- However, recent work [Fioravanti-Rossi, arXiv:2106.07600] - based on [Fioravanti-Rossi-Shu, arXiv:2004.10722] - has shed light on its **origin**, by "reversing it":

QQ/TQ/Bethe Ansatz

$\Downarrow$

Marchenko-like int. eq.

$\Downarrow$

Schrödinger ODEs

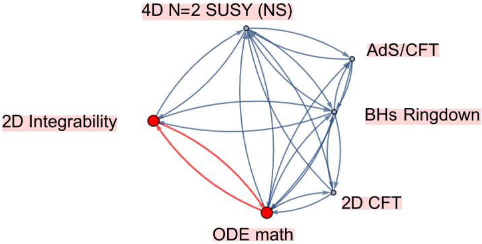


Figure: The Thinker (Rodin)

Three different physical theories, same mathematics!

$$-\frac{d^2}{dy^2}\psi(y) + \left[2e^{2\theta}\cosh y + P^2\right]\psi(y) = 0 \qquad \text{INTEGRABILITY (sd-Liouville)} \tag{24}$$

$$\frac{\hbar}{\Lambda_0} = \frac{\epsilon_1}{\Lambda_0} = e^{-\theta} \qquad \frac{u}{\Lambda_0^2} = \frac{1}{2}P^2 e^{-2\theta} \qquad \Downarrow$$

$$-\frac{\hbar^2}{2}\frac{d^2}{dy^2}\psi(y) + [\Lambda^2\cosh y + u]\psi(y) = 0 \qquad \text{N=2 GAUGE TH. } (SU(2)\ N_f = 0) \tag{25}$$

$$r = Le^{y/2} \quad \omega L = -2ie^{\theta} \quad P = \frac{1}{2}(l+2) \qquad \Downarrow \qquad \phi(r) = e^{y/2}\psi(y)$$

$$\frac{d^2\phi}{dr^2} + \left[\omega^2\left(1 + \frac{L^4}{r^4}\right) - \frac{(l+2)^2 - \frac{1}{4}}{r^2}\right]\phi(r) = 0 \qquad \text{BLACK HOLES PERT. (D3 brane)} \tag{26}$$

- We have shown this **triality** generally for Mathieu, Doubly Confluent Heun, Confluent Heun ODEs.

Black holes perturbation theory

- For perturbations of the **Schwarschild BH** metric $g^{(0)}_{\mu\nu}$, **Einstein equations**, after **linearization** and **gauge fixing** reduce to just a couple of much simpler PDEs:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{27}$$

$$\Downarrow \quad h_{\mu\nu} = g_{\mu\nu} - g^{(0)}_{\mu\nu} \Rightarrow (Z_l, Q_l)$$

1) for the **axial** perturbations (phase $(-1)^{l+1}$), the **Regge-Wheeler** equation

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right)\right] Q_l(t,x) = 0\,; \tag{28}$$

2) for the **polar** perturbations (phase $(-1)^l$), the **Zerilli** equation

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \left(1 - \frac{2M}{r}\right)\left(\frac{72M^3}{r^5\lambda(r)^2} + (\dots)\frac{12M}{r^3\lambda(r)^2}\left(1 - \frac{3M}{r}\right) + (\dots)\frac{1}{r^2\lambda(r)}\right)\right] Z_l(t,x) = 0, \tag{29}$$

where $\lambda(r) = l(l+1) - 2 + 6M/r$ and $r = r(x)$, with x tortoise coordinate.

From PDEs to ODEs

- The **Laplace transform** $\hat{\Psi}$ of $\Phi = Z_l \vee Q_l$ satisfies the non-homogeneous ODE

$$\hat{\Psi}(s,x) = \int_0^\infty e^{-st}\Phi(t,x)\,dt\,, \qquad \left\{-\frac{\partial^2}{\partial x^2} + U(x) + s^2\right\}\hat{\Psi}(s,x) = -\mathcal{I}(s,x)\,, \tag{30}$$

where $\mathcal{I}(s,x) = -s\Psi(t,x)|_{t=0} - \partial_t\Psi(t,x)|_{t=0}$. The corresponding **homogeneous** equation is the **ODE we will study**:

$$\left\{-\frac{\partial^2}{\partial x^2} + U(x) + s^2\right\}\Psi(s,x) = 0\,. \tag{31}$$

- We consider **solutions** of (31) **regular** for $x \to \pm\infty$, that is infinity and horizon (with $\mathrm{Re}\, s > 0$)

$$\Psi_+(s,x) \sim e^{-sx}\,, \qquad x \to +\infty\,, \qquad \Psi_-(s,x) \sim e^{sx}\,, \qquad x \to -\infty\,. \tag{32}$$

- The Laplace tr. $\hat{\Psi}$ is given by the **Green function** G as (with $x_< = \min(x',x)$, $x_< = \max(x',x)$)

$$\hat{\Psi}(s,x) = \int_{-\infty}^\infty G(s,x,x')\mathcal{I}(s,x')dx'\,, \qquad G(s,x,x') = \frac{1}{W[\Psi_-,\Psi_+]}\Psi_-(s,x_<)\Psi_+(s,x_>)\,. \tag{33}$$

Mathematical definition of QNMs

- Finally taking the anti-Laplace transform of $\hat{\Psi}$ and setting $s = i\omega$ we get the **original perturbation** $\Phi = Z_l \vee Q_l$ as [Nollert, Class.Quant.Grav. 16 (1999)]

$$\Phi(t,x) = \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} e^{st}\hat{\Psi}(s,x)\,ds \tag{34}$$

$$= \sum_n e^{i\omega_n t}\mathrm{Res}\left(\frac{1}{W(s)}\right)\Bigg|_{\omega_n}\int_{-\infty}^{\infty}\Psi_-(\omega_n,x_<)\Psi_+(\omega_n,x_>)\mathcal{I}(\omega_n,x')\,dx' + (\text{other contributions})\,.$$

- The **quasinormal modes (QNMs)** are the complex frequencies (real frequencies and damping times) ω_n, which corresponds also to **zeros of the ODE Wronskian**

$$W(\omega_n) \equiv W[\Psi_-,\Psi_+](\omega_n) = 0\,, \tag{35}$$

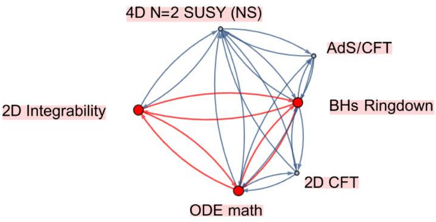
with $W[\Psi_-,\Psi_+] = \Psi_-(x)\Psi_+'(x) - \Psi_-'(x)\Psi_+(x)$.

Quasinormal modes of black holes as Bethe roots

- Through the ODE/IM formula $Q = W[\psi_+, \psi_-]$ we proved that the **Bethe root** (zero) condition on the Baxter's Q function

$$Q(\theta_n) = 0\,, \tag{36}$$

is equivalent to the mathematically rigorous definition of **quasinormal modes (QNMs)**! [Fioravanti-DG, arXiv:2112.11434]



Q has **QNMs as zeroes** and satisfies a **quantum Wronskian functional relation** which can be derived for **nearly any ODE**!

- In general the QQ-system **cannot** be directly inverted and solved as **integral equation**, rather it is necessary to construct **from** Q **the** Y **function** that solves the Y **system**.

Y function and quasinormal modes

- For example, for the **triad** [generalized Perturbed Hairpin/ $SU(2)$ $N_f = 2$ NS gauge th./ generalized extremal RN BHs] - *cf.* (3) - one can define a Y **function** as [Fioravanti-DG, arXiv:2112.11434; Fioravanti-DG-Shu, arXiv:2208.14031]

$$Y_{+,+}(\theta) = e^{i\pi(q_1-q_2)} Q_{+,+}(\theta) Q_{-,-}(\theta) \,. \tag{37}$$

- In terms of Y the QQ **system** is written as

$$e^{i\pi(q_1+q_2)} Q_{+,+}(\theta + \frac{i\pi}{2}) Q_{-,-}(\theta - \frac{i\pi}{2}) = 1 + Y_{+,-}(\theta) \,. \tag{38}$$

and for **QNMs** follow the alternative **quantizations conditions**

$$Y_{+,-}(\theta_n - i\pi/2) = -1 \qquad Y_{-,+}(\theta_n + i\pi/2) = -1 \,. \tag{39}$$

- From the QQ system it follows the Y **system** as

$$Y_{+,-}(\theta + \frac{i\pi}{2}) Y_{-,+}(\theta - \frac{i\pi}{2}) = [1 + Y_{+,+}(\theta)][1 + Y_{-,-}(\theta)] \,. \tag{40}$$

Thermodynamic Bethe Ansatz

- Defining as the **pseudoenergy**

$$\varepsilon_{\pm,\pm}(\theta) = -\ln Y_{\pm,\pm}(\theta)\,, \tag{41}$$

- the Y **system can be inverted in the Thermodynamic Bethe Ansatz (TBA)** for it

$$\begin{aligned}
\varepsilon_{\pm,\pm}(\theta) &= \frac{8\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2}e^{\theta} \mp i\pi(q_1 - q_2) - \int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\frac{\ln[1+\exp(-\varepsilon_{\pm,\mp}(\theta'))] + \ln[1+\exp(-\varepsilon_{\mp,\pm}(\theta'))]}{\cosh(\theta-\theta')} \\
\varepsilon_{\pm,\mp}(\theta) &= \frac{8\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2}e^{\theta} \mp i\pi(q_1 + q_2) - \int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\frac{\ln[1+\exp(-\varepsilon_{\pm,\pm}(\theta'))] + \ln[1+\exp(-\varepsilon_{\mp,\mp}(\theta'))]}{\cosh(\theta-\theta')} \,-
\end{aligned} \tag{42}$$

- **Technicality**: P enters as the $\theta \to -\infty$ boundary condition $\varepsilon_{+,+}(\theta, P) \sim 4P\theta + 2C(P, q_1, q_2)$, as $\theta \to -\infty$ (following from the perturbative expansion of ODE).

Gravity Y system and TBA

- In **gravity variables** Σ_k (combinations of four charges for **generalised RN extremal BHs**) the Y system reads
[Fioravanti-DG, arXiv:2112.11434; Fioravanti-DG-Shu, arXiv:2208.14031]

$$\begin{aligned} &Y(\theta + \frac{i\pi}{2}, -i\Sigma_1, -\Sigma_2, i\Sigma_3, \Sigma_4) Y(\theta - \frac{i\pi}{2}, -i\Sigma_1, -\Sigma_2, i\Sigma_3, \Sigma_4) \\ &= [1 + Y(\theta, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)][1 + Y(\theta, -\Sigma_1, \Sigma_2, -\Sigma_3, \Sigma_4)] \end{aligned} \tag{43}$$

- It can be inverted into the **TBA**

$$\begin{aligned} \varepsilon_\pm(\theta) &= [f_{0,+} \mp \frac{i\pi}{2}(\frac{\Sigma_1}{\Sigma_4^{1/4}} - \frac{\Sigma_3}{\Sigma_4^{3/4}})]e^\theta - \varphi * (\bar{L}_\pm + \bar{L}_\mp)(\theta) \\ \bar{\varepsilon}_\pm(\theta) &= [\bar{f}_{0,+} \pm \frac{\pi}{2}(\frac{\Sigma_1}{\Sigma_4^{1/4}} + \frac{\Sigma_3}{\Sigma_4^{3/4}})]e^\theta - \varphi * (L_\pm + L_\mp)(\theta) \end{aligned} \tag{44}$$

where we defined $\varepsilon_\pm(\theta) = -\ln Y(\theta, \pm\Sigma_1, \Sigma_2, \pm\Sigma_3, \Sigma_4)$, $\bar{\varepsilon}_\pm(\theta) = -\ln Y(\theta, \pm i\Sigma_1, -\Sigma_2, \mp i\Sigma_3, \Sigma_4)$, $L = \ln[1 + \exp\{-\varepsilon\}]$, $\varphi(\theta) = (2\pi \cosh(\theta))^{-1}$...

Quasinormal modes from TBA

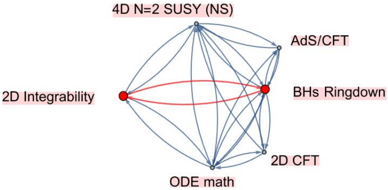
- The **QNMs condition** in gravity variables **reads alternatively as**

$$\bar{Y}_+(\theta_{n'} - i\pi/2) = -1\,, \quad Q_{+,+}(\theta_n) = 0\,,$$
$$\bar{\varepsilon}_+(\theta_n - i\pi/2) = -i\pi(2n+1)\,. \tag{45}$$

- Through the quantization condition on $\bar{\varepsilon}_+$ we can **actually numerically compute QNMs from TBA**.

n	l	TBA	Leaver	WKB $l \to \infty$
0	2	$1.47799 - 0.36814i$	$1.47789 - 0.36824i$	$1.494 - 0.3660i$
0	4	$2.68035 - 0.36664i$	$2.68035 - 0.36664i$	$2.689 - 0.3660i$
0	5	$3.27959 - 0.36641i$	$3.27959 - 0.36641i$	$3.287 - 0.3660i$
0	6	$3.87833 - 0.36629i$	$3.87833 - 0.36629i$	$3.884 - 0.3660i$
0	8	$5.07501 - 0.36615i$	$5.07501 - 0.36615i$	$5.080 - 0.3660i$
1	2	$1.21708 - 1.12174i$	$1.22004 - 1.12585i$	$1.494 - 1.0979i$
1	4	$2.54243 - 1.10434i$	$2.54246 - 1.10513i$	$2.689 - 1.0979i$
1	6	$3.78385 - 1.10090i$	$3.78380 - 1.10122i$	$3.885 - 1.0979i$
1	8	$5.00305 - 1.09963i$	$5.00300 - 1.09981i$	$5.080 - 1.0979i$

TABLE III. QNMs of (23) through (33) and other methods, with $\Sigma_1 = \Sigma_3 = 0.2$, $\Sigma_2 = 0.4$, $\Sigma_4 = 1$.



Comparison of QNMs computations methods

- Computing QNMs exactly is a non-trivial mathematical problem, it has been typically quite **laborious**, also because of their **few exact analytic characterizations**.
- The standard analytic method is the one with the **continued fractions** by Leaver is laborious and sometimes **sometimes not applicable** (in its original form).
- Application of $\mathcal{N} = 2$ **gauge theory** is a new analytic characterization, but requires a **nontrivial re-summation procedure** for $\omega_n \sim \Lambda_n \sim 1$.
- **Physics Informed Neural Networks (PINNs)** can approximate decentely complicated cases, but they bring **little new "human understanding"** of black hole physics / ODEs.

n	l	TBA	WKB
0	1	$0.89668 - 0.40069i$	$0.931 - 0.3946i$
0	2	$1.53080 - 0.39676i$	$1.551 - 0.3946i$
0	4	$2.78076 - 0.39525i$	$2.792 - 0.3946i$
0	8	$5.26781 - 0.39477i$	$5.274 - 0.3946i$
0	16	$10.2380 - 0.39472i$	$10.238 - 0.3946i$
0	32	$20.1738 - 0.39473i$	$20.165 - 0.3946i$
0	64	$40.0373 - 0.39473i$	$40.019 - 0.3946i$
0	128	$79.7581 - 0.39473i$	$79.729 - 0.3946i$
0	256	$159.185 - 0.39473i$	$159.147 - 0.3946i$

TABLE III. Comparison of QNMs of (23) with $\Sigma_1 = 0.1$, $\Sigma_2 = 0.2$, $\Sigma_3 = 0.3$, $\Sigma_4 = 1$, as obtained from TBA (33), (through (34) with $n' = 0$) and WKB approximation.

Our **integrability** method is **direct and exact**, but now we have developed it for just a **few models**.

Solving quantum Wronskians: D3 brane

- The scalar perturbation on the **D3 brane**, obeys **Mathieu ODE** - slide 26 - a particular case of DCHE (3), which is the same as $SU(2)$ $N_f = 0$ quantum SW curve.
- Through linear connection relations among solutions ψ_k one easily derives the **quantum Wronsian** (or QQ-system)

$$Q(\theta + i\pi/2, a)Q(\theta - i\pi/2, a) - Q(\theta, a)^2 = 1 \,. \tag{46}$$

- From the QQ **system** follows a **closed formula for** Q

$$Q(\theta, a) = \frac{\sinh\left[\frac{1}{\hbar} A_D(\theta, a)\right]}{\sin\left[\frac{2\pi}{\hbar} a\right]} \,, \quad \text{with} \quad A_D\left(\theta + i\pi/2, a\right) = A_D(\theta, a) - 2\pi i a \,, \tag{47}$$

We proved this and the following formulae purely from **ODE/IM and gauge theory**, but **alternative derivation** is possible through **AGT duality**

[Fioravanti-DG-Mahanta-Rossi,arXiv:25**.*****]

Solving QQ systems: extremal black holes

- We proved the QQ system for $N_f = 2$ gauge theory - *cf.* ODE (3), (22) - that is **extremal BHs**

$$Q\left(\theta + i\pi/2, -m_1, m_2\right) Q\left(\theta - i\pi/2, m_1, -m_2\right) - Q(\theta, -m_1, -m_2) Q(\theta, m_1, m_2) = e^{\frac{i\pi}{\hbar}(m_1 - m_2)} \tag{48}$$

is **equivalent to the shift relations**

$$A_D\left(\theta \pm i\pi/2, m_1, -m_2, a\right) = A_D\left(\theta, m_1, m_2, a\right) + h\left(m_2, a\right) \mp i\pi a \tag{49}$$

$$(-\partial_{m_1} - \partial_{m_2})\mathcal{F}_{inst}(\theta + i\pi/2, -m_1, -m_2, a) + (\partial_{m_1} + \partial_{m_2})\mathcal{F}_{inst}(\theta - i\pi/2, m_1, m_2, a) = 0$$

and similar others, with $h(m, a) = \ln \sqrt{\cos \frac{\pi}{\hbar}(a - m) \sec \frac{\pi}{\hbar}(a + m)}$.

- The **exact closed formula** for Q for $N_f = 2$ ensues

$$Q(\theta, m_1, m_2, a) = 2\pi \exp\left\{\frac{1}{\hbar}\sum_{k=1}^{2} \frac{\partial \mathcal{F}\left(\theta, m_1, m_2, a\right)}{\partial m_k}\right\} \frac{\sinh \frac{1}{\hbar} A_D\left(\theta, m_1, m_2, a\right)}{\sin \frac{2\pi}{\hbar} a}\,. \tag{50}$$

Solving QQ systems: general (Minkowski) black holes (I)

- We proved the first QQ system for $N_f = 3$ gauge theory - *cf.* ODE (5), (2), (23) - that is **Schwarschild BHs**

$$Q_1(\theta + i\pi, m_1, m_2, -m_3) Q_1(\theta, -m_1, -m_2, m_3) - Q_1(\theta, m_1, m_2, m_3) Q_1(\theta + i\pi, -m_1, -m_2, -m_3) = \frac{2i}{\hbar}(m_1 + m_2)\,, \tag{51}$$

is **equivalent to the shift relations**

$$A_D\left(\theta \pm i\pi, m_1, m_2, -m_3, a\right) = A_D\left(\theta, m_1, m_2, m_3, a\right) + h\left(m_3, a\right) \mp i\pi a\,, \tag{52}$$

$$(\partial_{m_1} + \partial_{m_2} - \partial_{m_3})\mathcal{F}_{inst}(\theta + i\pi, m_1, m_2, -m_3, a) + (-\partial_{m_1} - \partial_{m_2} + \partial_{m_3})\mathcal{F}_{inst}(\theta, -m_1, -m_2, m_3, a) = 0$$

and similar others.

- The **exact closed formula** for Q_1 ensues

$$Q_1\left(\theta, \mathbf{m}, a\right) = 2\pi \exp\left\{\frac{1}{\hbar}\sum_{k=1}^{3}\frac{\partial \mathcal{F}\left(\theta, \mathbf{m}, a\right)}{\partial m_k}\right\}\frac{\sinh \frac{1}{\hbar} A_D\left(\theta, \mathbf{m}, a\right)}{\sin \frac{2\pi}{\hbar} a}\,.$$

Solving QQ systems: general (Minkowski) black holes (II)

- We proved the second QQ system for $N_f = 3$ gauge theory, that is **Schwarschild BHs**

$$Q_2(\theta,-q_2,-q_1,q_3)Q_2(\theta,q_2,q_1,q_3) - Q_2(\theta,q_1,q_2,q_3)Q_2(\theta,-q_1,-q_2,q_3) = q_1^2 - q_2^2 \tag{53}$$

is equivalent to the shift relation

$$\partial_{m_1}\mathcal{F}_{inst}(\theta,m_1,m_2,m_3,a) - \partial_{m_1}\mathcal{F}_{inst}(\theta,-m_1,-m_2,m_3,a) = 0\,. \tag{54}$$

and similar others

- The **exact closed formula** for Q_2 ensues

$$Q_2\left(\theta,\mathbf{m},a\right) = \exp\left\{\frac{2}{\hbar}\frac{\partial\mathcal{F}\left(\theta,\mathbf{m},a\right)}{\partial m_1}\right\}\,. \tag{55}$$

[Fioravanti-DG-Mahanta-Rossi,arXiv:25**.*****]

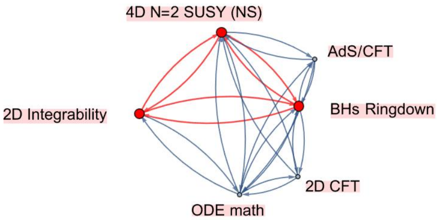
Proof of new ($\mathcal{N} = 2$) gauge - gravity duality

- Since proved a relation between **at least one** Q **function and the gauge dual period**

$$Q(\theta, \mathbf{m}, a) = \exp\left\{\frac{1}{\hbar}\sum_{k=1}^{N_f}\frac{\partial \mathcal{F}(\theta, \mathbf{m}, a)}{\partial m_k}\right\}\frac{\sinh\frac{1}{\hbar}A_D(\theta, \mathbf{m}, a)}{\sin\frac{2\pi}{\hbar}a}, \tag{56}$$

  from our formalism **it follows immediately that QNMs are given by quantization conditions on the gauge periods** *cf.* (11)

$$Q(\theta, \mathbf{m}, a) = 0 \quad \Longrightarrow \quad A_D(\theta, \mathbf{m}, a) = 2i\pi\hbar\, n\,. \tag{57}$$

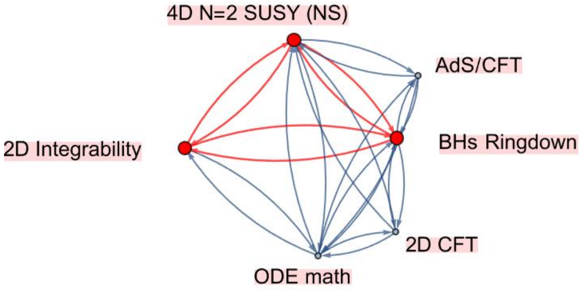


- (57) was previously found heuristically and allowed:
  - a **novel analytic characterization of QNMs**;
  - to **find new results for the BHs theory** (by many other authors);
  - an **unexpected application of Supersymmetry**.

Conclusions

Summary

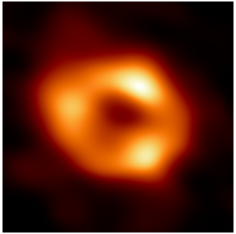
- We have shown **BHs perturbation theory** has a natural connection with
  - 4D $\mathcal{N} = 2$ **NS SUSY gauge theory**;
  - 2D **integrable models**.
- In particular **ODE/IM correspondence** allowed
  - to **find new results** an all sides, at the **exact level**;
  - a **new exact method of computation of QNMs**: the Thermodynamic **Bethe Ansatz** (TBA);
  - a (mathematical) **proof** of the **SW-QNM correspondence**.



Future developments

We notice that **much of the BH theory seems to go in parallel to the ODE/IM correspondence**, beyond the determination of QNMs!

- $\mathcal{N} = 2$ gauge theories and BHs constitute **novel fields of application of integrability** and many future developments could be pursued:
  - generalization to **more BHs models**;
  - extension to **other BHs observables**;
  - addressing **theoretical problems in BH physics**;
  - tackling **non-linear** perturbation theory;
  - **connection to other fields** (like holography).



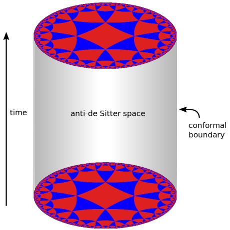


Figure: (Wikicommons/Herzog,Dunkel)

Thank you for your attention!

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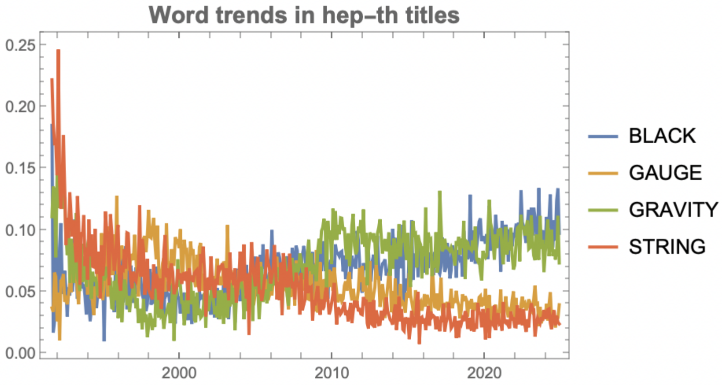

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