From Integrability to Supersymmetry and Black Holes

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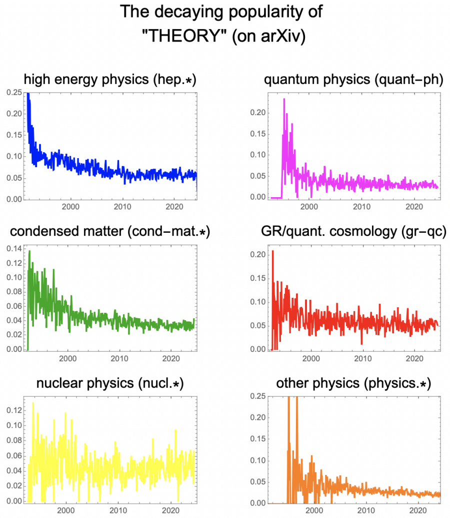
Based on: arXiv:24**.*****, arXiv:2208.14031, arXiv:2112.11434, arXiv:1908.08030

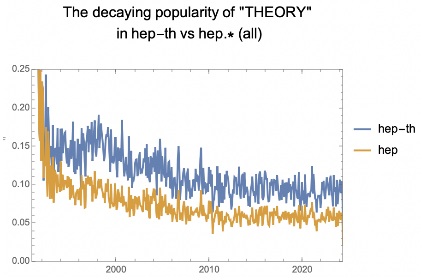
with D. Fioravanti, R. Mahanta, M. Rossi, H. Shu

Introduction
Evolution of Theoretical Physics

A bleak outlook for “theory”?



Analysing titles of **all arXiv articles**, we find the share of **popularity of the word “theory”** is decreasing, across all physics fields.

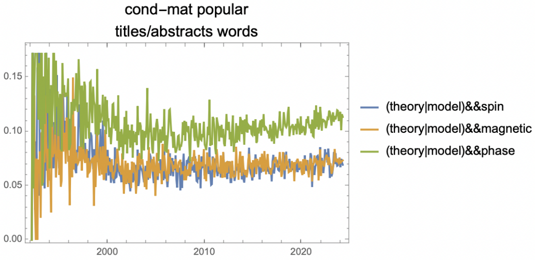


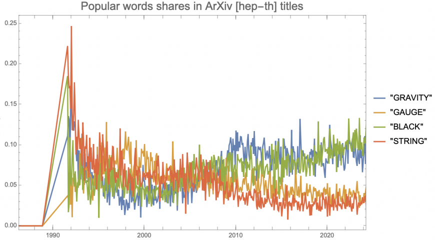
In high energy theory it is perhaps even clearer.

Theory on/and what?

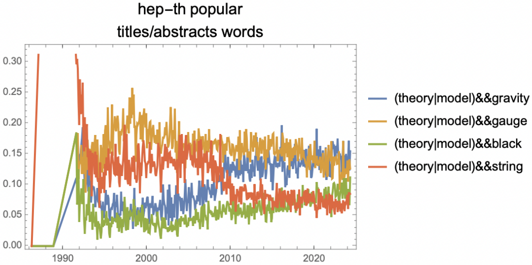
Looking more carefully at: most popular words in titles (below); their combinations in abstracts (right), we can actually identify a **“phase transition”** (quoting Prof. V. Cardoso) in hep-th.

1	quantum	44 541
2	model	24 312
3	spin	23 679
4	magnetic	21 930
5	phase	19 334
6	2	18 624
7	dynamics	17 086
8	effect	15 076
9	systems	14 949
10	theory	14 355

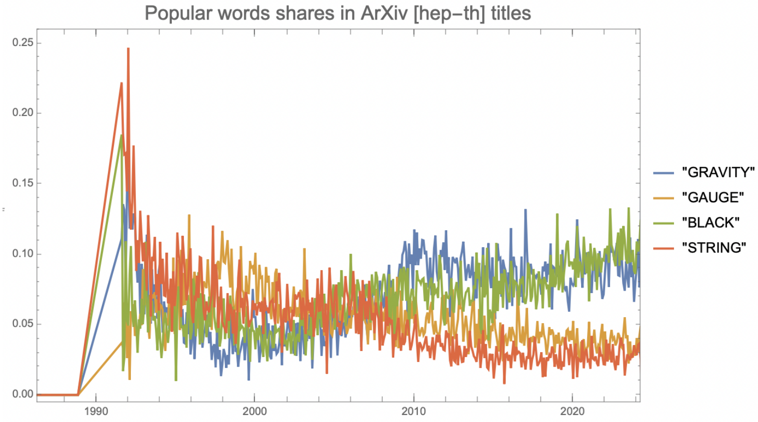




1	theory	18 039
2	quantum	16 295
3	gravity	12 189
4	black	11 992
5	field	11 989
6	model	9666
7	theories	8685
8	gauge	8681
9	n	8596
10	string	7379



Insights from ArXiv data?



Check out the **notebook** of my little arXiv data analysis [work in progress]:

`https://github.com/Daniele-Gregori/ArXiv-Articles-Data-Analysis`

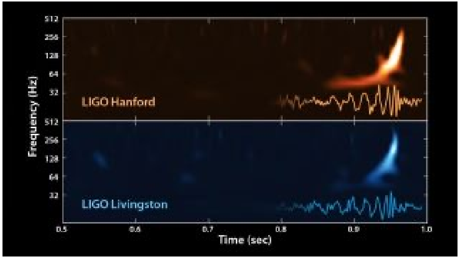
How to **interpret** the **general shift** away from "string" / "gauge" towards "gravity" / "black"?

Introduction
A new era for Black Hole Physics

"A new window to the universe"

In the last few years, **gravitational waves** detections and **black hole imaging** have opened the way for **gravitational phenomenology**.

The new observations promise to revolutionise not only astrophysics but even **fundamental physics**, allowing the search of **deviations from general relativity (GR)** in previously unexplored systems and regimes.



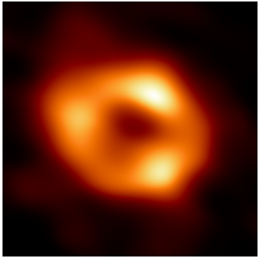


Figure: LIGO, Event Horizon

Since **BHs are the simplest and most perfect objects** in gravity, one could ask

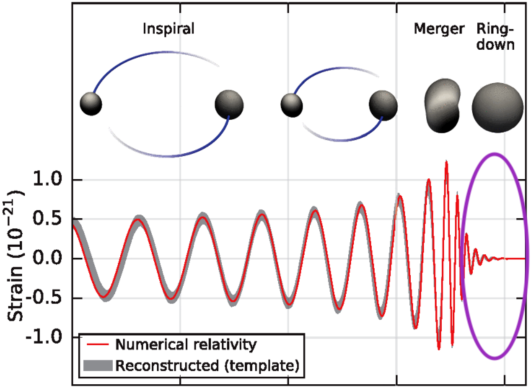
Could BHs hold the same surprises that the electron and the hydrogen atom did when they started to be experimentally probed? *[Cardoso-Pani, arXiv:1709.01525]*

Colliding BHs and quasinormal modes

A **black hole collision** can be divided in 3 phases: **inspiral, merger** and **ringdown**.

QNMs can be observed as the complex frequencies of the **damped oscillations of spacetime** in the final - **ringdown** - phase of black holes (BHs) merging.

[Nollert, Class.Quant.Grav. 16 (1999); Berti-Cardoso-Starinets, arXiv:0905.2975]



Black holes perturbations can be expressed in terms of solutions of **ODEs shared with other physics fields!**

The need of new methods

- Even if QNMs for GR BHs are already known, it is still an open problem to compute them in **modified gravity**. Specifically, the common methods are:
  - **Continued fraction** method: exact but laborious;
  - **WKB approximation**: easy but rough, asymptotic, to be resumed;
  - **Numeric simulations**: case by case, sometimes unstable, with little insight.

Therefore developing **new analytic characterizations and exact numerical methods** for QNMs is very important.

- QNMs also present **theoretical problems even in GR** like their instability under small perturbations of the background.

$$\frac{1}{4x^2(x-1)^3}\sum_{n=0}^{2}\sum_{m=0}^{2-n} A_{n,m}\, g^{(n)}(x)\,(i\,\Omega)^m = 0\,. \qquad (21)$$

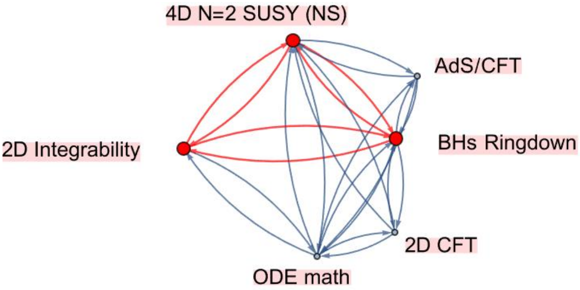
Here $g^{(n)}(x)$ is the n-th derivative of the function $g(x)$. The real coefficients $A_{n,m}(x)$ are

$$\begin{aligned}
A_{0,0} =& -x^2(x-1)^2\,[4(\ell-1)(\ell+2)\mathcal{M}_0\,\mathcal{S}_0^3\\
&-x(8\mathcal{N}_0\,\mathcal{S}_0' + x\mathcal{N}_0'\,\mathcal{S}_0' + 2x\mathcal{N}_0\,\mathcal{S}_0'')\,\mathcal{M}_0\,\mathcal{W}_0\,\mathcal{S}_0\\
&+x(2\mathcal{S}_0 - x\mathcal{S}_0')\,\mathcal{M}_0'\,\mathcal{N}_0\,\mathcal{W}_0\,\mathcal{S}_0\\
&+3x^2\mathcal{M}_0\,\mathcal{N}_0\,\mathcal{W}_0\,\mathcal{S}_0'^2\\
&+2(4\mathcal{N}_0 + x\mathcal{N}_0')\,\mathcal{M}_0\,\mathcal{W}_0\,\mathcal{S}_0^2]\,\mathcal{P}_0\\
A_{0,1} =& -2x^2\,[2(\delta + (x^2-2x)(\gamma+\delta))\mathcal{M}_0\,\mathcal{N}_0\\
&+(x-1)B(x)(\mathcal{M}_0\,\mathcal{N}_0)']\,\mathcal{P}_0\,\mathcal{W}_0\,\mathcal{S}_0^2\,,\\
A_{0,2} =& +4\,[B^2(x)\,\mathcal{M}_0\,\mathcal{N}_0\,\mathcal{P}_0 - (x-1)^2\,\mathcal{S}_0]\,\mathcal{W}_0\,\mathcal{S}_0^2\,,\\
A_{1,0} =& +2x^3(x-1)^2\,[x(\mathcal{M}_0\,\mathcal{N}_0)' + 4\mathcal{M}_0\mathcal{N}_0]\,\mathcal{P}_0\,\mathcal{W}_0\,\mathcal{S}_0^2\,,\\
A_{1,1} =& -8x^2(x-1)\,B(x)\,\mathcal{M}_0\,\mathcal{N}_0\,\mathcal{P}_0\,\mathcal{W}_0\,\mathcal{S}_0^2\,,\\
A_{2,0} =& +4x^4(x-1)^2\,\mathcal{M}_0\,\mathcal{N}_0\,\mathcal{P}_0\,\mathcal{W}_0\,\mathcal{S}_0^2\,,
\end{aligned}$$

Figure: ODE for axial perturbation in Einstein scalar Gauss-Bonnet theory [arXiv:2404.11583]

Plan of the talk

1. Introduction
   - Evolution of Theoretical Physics
   - A new era for Black Hole Physics
2. The search for new methods
   - Artificial and natural intelligence
   - Rescuing supersymmetry
3. A more radical method
   - Introduction to Bethe Ansatz
   - ODE/IM correspondence
   - QNMs as Bethe roots
   - Solving QQ-system of gauge theories
4. Outlook
   - Conclusions and future developments

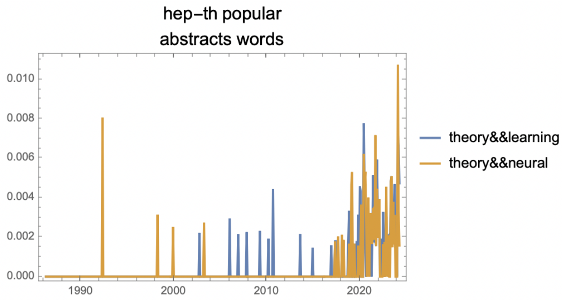


The search for new methods

Artificial and natural intelligence

Where has "theory" gone?

Recent years have seen an upsurge of applications of **Machine "learning"** and **"Neural" networks** in theoretical physics...



Even if computer science (CS) seems not usually inclined to "theory"!

arXiv CS.*
title words

1	learning	81 664
2	using	44 542
3	networks	42 908
4	neural	33 557
5	deep	30 129
6	data	28 620
7	models	26 383
8	network	26 329
9	detection	24 460
10	model	24 009

...105 words before "theory"...

101	clustering	4936
102	federated	4902
103	understanding	4901
104	diffusion	4824
105	channel	4740
106	$\boxed{\text{theory}}$	4739
107	functions	4721
108	local	4717
109	planning	4712
110	representations	4700

Actually, AI can steal my job!

arXiv:2404.11583v1 [gr-qc] 17 Apr 2024

Quasinormal Modes in Modified Gravity using Physics-Informed Neural Networks

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In this paper, we apply a novel approach based on physics-informed neural networks to the computation of quasinormal modes of black hole solutions in modified gravity. In particular, we focus on the case of Einstein-scalar-Gauss-Bonnet theory, with several choices of the coupling function between the scalar field and the Gauss-Bonnet invariant. This type of calculation introduces a number of challenges with respect to the case of General Relativity, mainly due to the extra complexity of the perturbation equations and to the fact that the background solution is known only numerically. The solution of these perturbation equations typically requires sophisticated numerical techniques that are not easy to develop in computational codes. We show that physics-informed neural networks have an accuracy which is comparable to traditional numerical methods in the case of numerical backgrounds, while being very simple to implement. Additionally, the use of GPU parallelization is straightforward thanks to the use of standard machine learning environments.

I. INTRODUCTION

Black hole characteristic ringing has been long considered one of the ultimate tests of the laws of gravity at extreme curvature [1–3]. Detecting at least two quasinormal mode (QNM) frequencies can already put severe constraints on any deviation from general relativity (GR) [4, 5] or the other way around – it can provide an undoubted signature of a failure of Einstein's theory. Moreover, with the direct detection of gravitational waves by the Advanced LIGO and Advanced Virgo detectors [6, 7] from compact binary coallescences [8–11] and with the gradual upgrades of their sensitivities, observing high-precision post-merger black hole ringing becomes realistic [12–19]. This is especially true for the next generation of gravitational-wave detectors such as LISA, the Einstein Telescope [20], and the Cosmic Explorer [21].

There are a number of challenges in properly interpreting the gravitational wave signal emitted during such characteristic ringing. This includes for example the fact that the post-merger signal is not a clear one and it will be contaminated with the nonlinear part of the actual merger of two black holes [22, 23] igniting nonlinear modes in the spectrum [24–26]. Overtones or higher harmonics can also play an important role [27–29]. Last very involved since the equations are stiff [32, 33]. Subtle adjustments of the numerical codes are required and the solution is typically done case by case [34]. The picture worsens if we want to extend these results to the case of rapid rotation since in most of the realistic theories beyond GR separation of variables is not possible and one ends up with a system of partial differential equations instead.

To address the aforementioned obstacles, several well-designed approximations have been developed. These typically include an expansion in the coupling parameter for the beyond-GR theory, assuming that the GR corrections are small, and an expansion in terms of the rotational parameter [35–42]. Sophisticated new methods for calculating the QNM frequencies using spectral decomposition are also being developed [43–45], showing promising potential but still posing a number of obstacles when dealing with beyond-GR theories. These approaches typically require important adjustments and the efforts behind each methodology/code development can be substantial. For example, the choice of a spectral basis can be quite critical for controlling the accuracy of spectral methods [43–45]. Moreover, to avoid the contamination of the numerical solution with an ingoing wave component when performing direct numerical in-

"If one's doing something like solving a single equation (say, ODE) precisely, AI probably won't be the best tool. But if one's got a **big collection of equations** *[...]* **AI may successfully be able to give a useful "rough estimate"** *of what will happen, even when traditional methods would get utterly bogged down in details." (S. Wolfram)*

I will not let PINNs steal my job!

Physics Informed Neural Networks (PINNs) are an interesting approach to try... but it may not be always the best and in any case they bring us **little new "human understanding"** of BHs or ODEs...

arXiv CS.*
title words

1	learning	81664
2	using	44542
3	networks	42908
4	neural	33557
5	deep	30129
6	data	28620
7	models	26383
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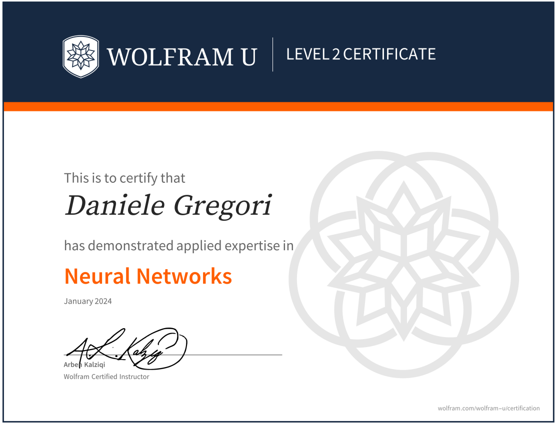


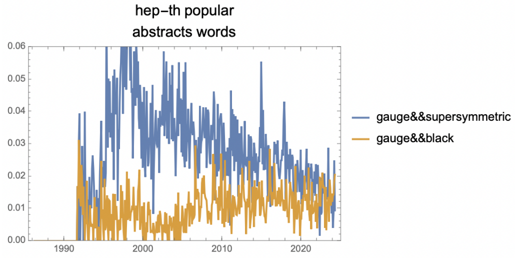
The search for new methods

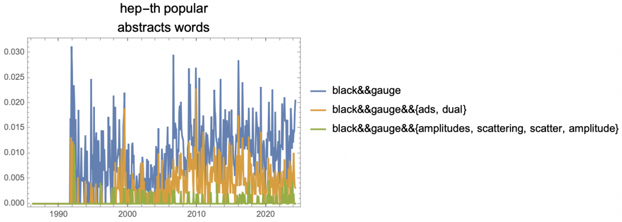
Rescuing supersymmetry

To the rescue of supersymmetry?

We see an increasing of article abstracts talking simultaneously of **gauge theory and black holes**.

Actually, it is mostly due to **AdS/CFT** and NOT to the well known community of **scattering amplitudes**, modeling the inspiral phase of black holes as particle collision.



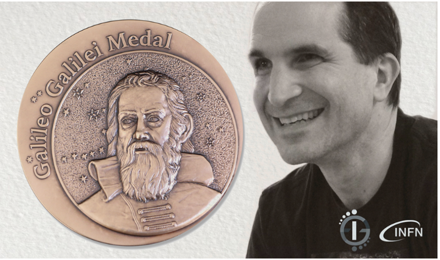


An even smaller community works differently with **Seiberg-Witten theory** on (mostly) the ringdown phase...

Are AdS black holes more **physical** than supersymmetric particles?

Galileo and Maldacena

At least twice in my short scientific life - during formal meetings at Bologna University - **I have been publicly attacked for proposing the study of ($\mathcal{N} = 2$) supersymmetry**...



Similarly - according to rumours - someone complained for the Galileo Galilei award to Prof. Maldacena, on the basis of **"not being an experimental physicist, like Galileo was"**...

...but in the *Saggiatore* we read also "the book of nature is written in..."

“The book of nature is written in mathematical language!”

Generally in both **General Relativity** and **modified gravity**, BHs perturbations obey different kind of ODEs.

- Perturbations of **asymptotically AdS_5 BHs** obey **Heun** equation

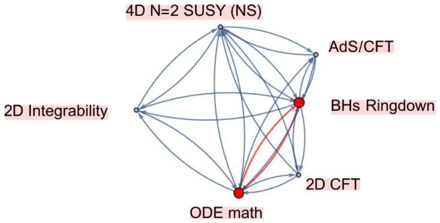
$$w''(z) + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\alpha+\beta-\gamma-\delta+1}{z-a}\right) w'(z) + \frac{\alpha\beta z - q}{z(z-1)(z-a)} w(z) = 0\,. \qquad (1)$$

- for 4D **asymptotically flat BHs**, **Confluent Heun** equation (CHE)

$$w''(z) + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon\right) w'(z) + \frac{\alpha z - q}{z(z-1)} w(z) = 0\,. \qquad (2)$$

- for 4D **Extremal** flat BHs, **Doubly Confluent Heun** eq. (DCHE)

$$w''(z) + \left(\frac{\delta}{z^2} + \frac{\gamma}{z} + 1\right) w'(z) + \frac{\alpha z - q}{z^2} w(z) = 0\,. \qquad (3)$$



- Caveat: **rotating BHs** involve **coupled ODEs** (separable PDEs)!
- The inspiral phase is governed by the **non-homogenous** same ODEs!

$\mathcal{N} = 2$ SUSY primer

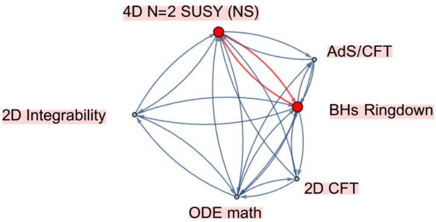
- **Seiberg-Witten (SW) theory** is the effective theory for $\mathcal{N} = 2$ SUSY gauge theory. To compute **instanton contributions**, it is convenient to deform spacetime in terms of two parameters $\epsilon_1, \epsilon_2 \in \mathbb{C}$ (Ω**-background**).
  [Seiberg-Witten, hep-th/9407087; Nekrasov, hep-th/0206161]
- If $\epsilon_2 \to 0$ - **Nekrasov-Shatashvili (NS) limit** - the prepotential $\mathcal{F}$ can obtained from monodromies of ODEs of Schrödinger time-independent type with $\epsilon_1 = \hbar \neq 0$ (for $SU(2)$)

$$-\hbar^2 \frac{d^2}{dy^2}\psi(y) + \left[U(y; \mathbf{m}, \Lambda) - u\right]\psi(y) = 0\,, \tag{4}$$

[Nekrasov-Shatashvili, arXiv:0908.4052]

No jokes: Schwarshild black hole!

The quantum SW curve for $N_f = 3$ $SU(2)$ **gauge th.** can be mapped to the Regge-Wheeler equation for linear **perturbations of the Schwarschild BH!**



$$-\hbar^2\frac{d^2}{dy^2}\psi(y)+\Big\{e^{2y}\Lambda_3\left(4(m_1-m_2)^2\right)$$
$$+\left(\Lambda_3^2-24\Lambda_3 m_3+64u\right)+4e^{y}\sqrt{\Lambda_3}\left(-2\hbar^2+8m_1m_2+\Lambda_3m_3-8u\right)$$
$$+4e^{-y}\sqrt{\Lambda_3}\left(8m_3-\Lambda_3\right)+4\Lambda_3e^{-2y}\Big\}\frac{1}{16\left(\sqrt{\Lambda_3}e^{y}-2\right)^2}\psi(y)=0 \tag{5}$$

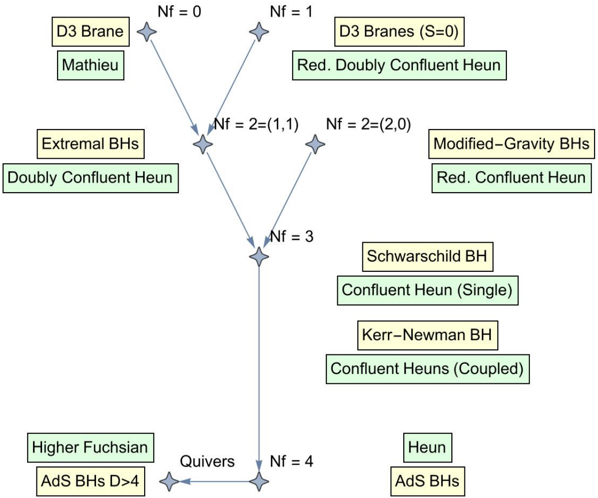
$$\Lambda_3=-16iM\omega\,,\qquad u=l(l+1)-8M^2\omega^2+\frac{1}{4}\,,\qquad \Big\Downarrow \qquad r=\frac{4M}{\Lambda_3}e^{-y}\,,$$

$$\hbar=1\,,\qquad m_1=s-2iM\omega\,,\qquad m_2=-s-2iM\omega\,,\qquad m_3=-2iM\omega\,,$$

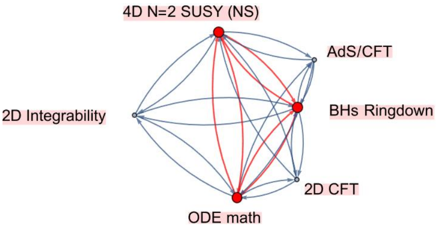
$$\left[f(r)\frac{d}{dr}f(r)\frac{d}{dr}+\omega^2-f(r)\left[\frac{l(l+1)}{r^2}+(1-s^2)\frac{2M}{r^3}\right]\right]\phi(r)=0\,, \tag{6}$$

$$ds^2=-f(r)dt^2+f(r)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2)\,,\quad \text{with } f(r)=1-2M/r$$

General picture of new gauge - gravity correspondence



$SU(2)$ $\mathcal{N} = 2$ (NS) gauge theories, Heun ODEs and BHs models.



[Aminov-Grassi-Hatsuda, arXiv:2006.06111, Bianchi-Consoli-Grillo-Morales, arXiv:2109.09804]

Nekrasov-Shatashvili instanton series

- The NS prepotential can be obtained as a **series in the instanton coupling** Λ

$$\mathcal{F}(\Lambda, a, \hbar) = \mathcal{F}^{pert}(a, \hbar) + \sum_{n=1}^{\infty} \mathcal{F}_n^{inst}(a, \hbar)\Lambda^{(4-N_f)n} . \tag{7}$$

- The **gauge period** a is recovered from the **ODE moduli parameter** u through

$$u = \Lambda \frac{\partial \mathcal{F}(\Lambda, a, \hbar)}{\partial \Lambda} . \tag{8}$$

- Also the **gauge dual instanton period** A_D is defined as

$$A_D = \frac{\partial \mathcal{F}(\Lambda, a, \hbar)}{\partial a} \tag{9}$$

and in the $\hbar \to 0$ SW limit, $A_D \simeq \frac{1}{\hbar} A_D^{(0)}$, with $A_D^{(0)} = a_D^{(0)} + 2\pi i a^{(0)}$.

Nice analytical expressions!

- For example for $SU(2)$ gauge theory with $N_f = 3$ matter flavours, at **second order** we get:

$$\mathcal{F}^{inst}(\Lambda_3, m_1, m_2, m_3, a, \hbar) = -\frac{m_1 m_2 m_3}{4(4a^2-\hbar^2)}\Lambda_3$$
$$+\Bigg[\frac{-256a^8 - 256a^6\left(m_3^2+m_1^2+m_2^2-2\hbar^2\right) + 96a^4\left(2m_3^2\hbar^2 + 2m_2^2\left(4m_3^2+\hbar^2\right) + 2m_1^2\left(4m_3^2+4m_2^2+\hbar^2\right) - 3\hbar^4\right)}{8192\left(a^2-\hbar^2\right)\left(4a^2-\hbar^2\right)^3}$$
$$+\frac{-16a^2\left(3m_3^2\hbar^4 + 3m_2^2\left(8m_3^2\hbar^2+\hbar^4\right) + m_1^2\left(8m_2^2\left(10m_3^2+3\hbar^2\right) + 3\left(8m_3^2\hbar^2+\hbar^4\right)\right) - 4\hbar^6\right)}{8192\left(a^2-\hbar^2\right)\left(4a^2-\hbar^2\right)^3}$$
$$+\frac{4m_3^2\hbar^6 + 4m_2^2\hbar^6 + 48m_3^2m_2^2\hbar^4 + 4m_1^2\hbar^2\left(12m_3^2\hbar^2 - 4m_2^2\left(28m_3^2-3\hbar^2\right) + \hbar^4\right) - 5\hbar^8}{8192\left(a^2-\hbar^2\right)\left(4a^2-\hbar^2\right)^3}\Bigg]\Lambda_3^2 + O(\Lambda_3^3)\,. \tag{10}$$

- The computation algorithm is based on **combinatorics on Young diagrams**

Nice **analytical** expressions...but the terms are infinite and the computation time for higher orders **grows exponentially**!

Rescued by supersymmetry?

It was found heuristically that **quasinormal modes (QNMs) are given by quantization conditions on the gauge periods**

$$A_D(\Lambda, \mathbf{m}, u) = 2i\pi\hbar\, n \quad \text{for } n \in \mathbb{N} \qquad (11)$$

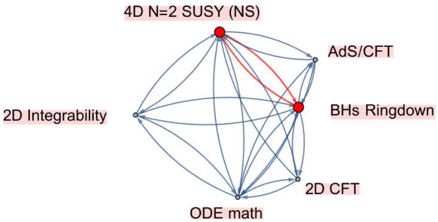
$$\Downarrow$$

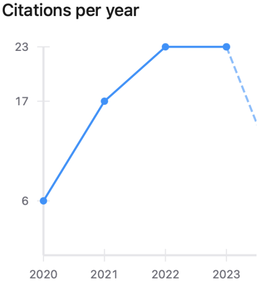
$$\text{QNMs} \quad \omega_n \propto \Lambda$$

[Aminov-Grassi-Hatsuda, arXiv:2006.06111]

A small field is growing on this result, since:

- qSW theory turn out to be a **very effective method for QNMs computation**
- it also provided **new theoretical results**.

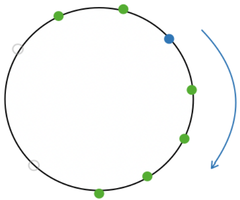


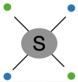


A more radical method

Introduction to Bethe Ansatz

Bethe ansatz





$e.g. = \frac{p_l - p_m - i}{p_l - p_m + i}$

$$e^{ip_l L} \prod_{m=1, m\neq l}^{L} S(p_l, p_m) = 1$$

free phase scattering phase

Hans Bethe first in 1931 devised a method to **completely solve the Heisenberg spin chain** model by means of a **finite closed system of algebraic equations**.

This follows from the (Bethe) ansatz "propagation of a test particle and its scattering on the others equals unity."

Valid for finite size (periodic) spin chains, not field theories!

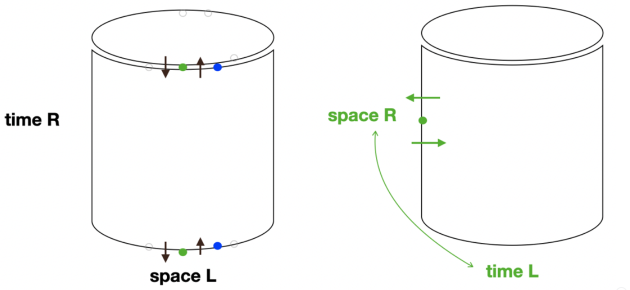
From condensed matter to string theory

At that time, Bethe himself did not pursue farther his innovative approach. Nevertheless, **in later decades many other physical systems** were completely solved in the same way...

The **same mathematics** appears in physical systems ranging from condensed matter, quantum many body, optics to gauge theory, string theory and AdS/CFT!

The method itself (coordinate BA) was developed into

- Nested BA
- Asymptotic BA
- Algebraic BA
- Functional BA
- Thermodynamic BA (TBA)
- Quantum Spectral Curve (QSC)



Ice model and continuum limit

- For **twisted 6 vertex (ice) model**, with spectral parameter ν and anisotropy η BA reads

$$(-1)^n \prod_{j=1}^{n} \frac{q(\nu_k - 2i\eta)}{q(\nu_k + 2i\eta)} = -e^{-2i\phi} \frac{(\sin(\eta + i\nu_k))^N}{(\sin(\eta - i\nu_k))^N} \tag{12}$$

in terms of $q(\nu) = \prod_{l=1}^{n} \sinh(\nu - \nu_l)$.

- In the **continuum limit** $N \to \infty$

$$\prod_{l=1}^{\infty} \left(\frac{E_l - \omega^2 E_i}{E_l - \omega^{-2} E_i} \right) = -e^{-2i\phi} \tag{13}$$

in terms of $E_i \sim e^{2\nu_i} N^{4\eta/\pi}$, $\omega = -e^{-2i\eta}$

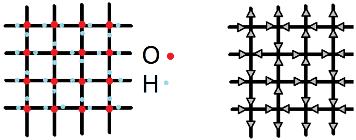


Figure: (Wikicommons/Moali)

- In this limit the BA follows from (equalling zero) **Baxter's** TQ **relation**

$$T(E)Q_\pm(E) = e^{\mp 2\pi i p} Q_\pm(\omega^{-2}E) + e^{\pm 2\pi i p} Q_\pm(\omega^2 E) \tag{14}$$

where $p \sim \sqrt{\Delta}$, $\omega^2 = \exp(-4i\eta)$

$$Q_+(E) = \prod_{i=1}^{\infty} \left(1 - \frac{E}{E_i}\right) \tag{15}$$

On quantum integrability

- The hallmark of quantum integrability is the presence of **infinite integrals of motion commuting** with each other
$$[\mathbf{I}_n, \mathbf{I}_m] = 0\,. \tag{16}$$
- The IM I_{2n-1} are also asymptotic expansion coefficients, for large spectral parameter $\lambda = e^{\theta}$, of the **Baxter's** Q, T **operators**
$$\ln \mathbf{Q}(\theta) \simeq -C_0 e^{\theta} - \sum_{n=1}^{\infty} e^{-n\theta} C_n \mathbf{I}_n \qquad \theta \to +\infty\,, \tag{17}$$
$$[\mathbf{Q}(\theta), \mathbf{T}(\theta)] = 0\,, \tag{18}$$
[Bazhanov-Lukyanov-Zamolodchikov, hep-th/9412229, hep-th/9604044]
- Their eigenvalues, the Q, T **functions** satisfy **functional relations**, like **quantum wronskian** QQ or **Baxter's** TQ, which given suitable **asymptotic behaviours** have **unique solutions**.

A more radical method
ODE/IM correspondence

ODE/IM correspondence essentials

- The first ODE/IM instance found was for the **anharmonic oscillator ODE**, with $\alpha \geq -1$

$$-\frac{d^2}{dx^2}\psi(x) + \left(x^{2\alpha} + \frac{l(l+1)}{x^2} - E\right)\psi(x) = 0\,. \tag{19}$$

- The **spectral determinants** are also computed as **Wronskians**

$$Q^+(E,l) = Q_0^+ \prod_{k=0}^{\infty}\left(1 - \frac{E}{E_{2k}}\right) = W[\psi_{+,0}, \psi_{-,0}] \text{ with } \begin{cases} \psi_{+,0}(x) \to 0\,, & x \to \infty \\ \psi_{-,0}(x) \to 0\,, & x \to 0 \end{cases}. \tag{20}$$

- $Q^+(E,l)$ can be **computed** as **integral** of the solution $\mathcal{P}(x) = -i\frac{d}{dx}\ln\psi(x)$ of the **Riccati** ODE equivalent to (19)

$$Q^+(E,l) = \lim_{x\to 0}[x^l\psi_{+,0}(x)] = \exp\left\{i\int_0^\infty [\mathcal{P}(x) - \text{regulariz.}\,]\,dx\right\}. \tag{21}$$

- Through **discrete symmetries** of the ODE, we can define other solutions in $\mathbb{C}$, with $\omega = e^{\frac{i\pi}{\alpha+1}}$

$$\Omega_+ : x \to \omega^{-1} x\,, \quad E \to \omega^2 E \quad \Longrightarrow \quad \psi_{+,k} \equiv \Omega_+^k \psi_{+,0} = \psi_{+,0}(\omega^{-k} x, \omega^{2k} E, l)\,, \tag{22}$$

so we can write **lateral connection relations** in terms **Stokes multipliers** $T_{k/2}$

$$\psi_{-1}(x) = T_{k/2}(E,l)\psi_{k-1}(x) + \tilde{T}_{k/2}(E,l)\psi_k(x) \quad \Longrightarrow \quad T_{k/2}(E,l) = W[\psi_{-1},\psi_k]\,. \tag{23}$$

- Similarly defining $\psi_{-,1}(x,E,l) \equiv \psi_{-,0}(x,E,-l-1)$ and $Q^-(E,l) = Q^+(E,-l-1)$ from **central connection relations** (between $\psi_{-,k}$ and $\psi_{+,k}$) we obtain QQ **system**

$$\omega^{1/2} Q^+(\omega^{-1}E) Q^-(\omega E) - \omega^{-1/2} Q^+(\omega E) Q^-(\omega^{-1} E) = 2i\,. \tag{24}$$

- Similarly also TQ and T systems.
- $Q^\pm$, $T_{k/2}$ realize **vacuum** Q, T **functions** of **Scaling 6-vertex IM!** [Dorey-Tateo, hep-th/9812211, Bazhanov-Lukyanov-Zamolodchikov, hep-th/9812247]

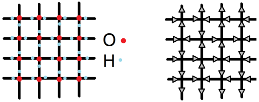
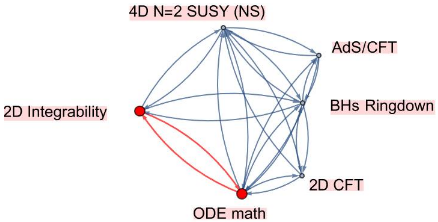


Figure: (Wikicommons/Moali)

The ODE/IM correspondence

- The **ODE/IM correspondence** is an approach to integrability, in which the Q, T functions and all the (QQ, TQ...) **integrable structures** of some Integrable Model (IM) can be **derived from an Ordinary Differental Equation** (ODE), even without knowing a priori the corresponding IM. [Dorey-Tateo, hep-th/9812211, Bazhanov-Lukyanov-Zamolodchikov, hep-th/9812247, Dorey-Dunning-Tateo, hep-th/0703066]

ODE	IM
Spectral determinant	Q
Stokes multipliers	T_k
Central Connect. rels.	QQ relation
Lateral Connect. rels.	TQ, T-syst
ODE parameters	IM parameters
ODE ind. var. x	... cf. below



ODE/IM allows to apply integrability to very different physical theories!

Other ODE/IM examples: Liouville

- We can reach the regime $\alpha < -1$, by a formal change of variable, leading to the ODE for the to **Liouville model** , that is CFT central charge $c = 1 + (b + 1/b)^2 \geq 1$

$$-\frac{d^2}{dy^2}\psi(y) + \left[e^{2\theta}(e^{y/b} + e^{-by}) + P^2\right]\psi(y) = 0\,, \tag{25}$$

which reduces to (modified) Mathieu equation for $b = 1$ ($c = 25$).
[Fioravanti-DG, arXiv:1908.08030]

- Liouville model is
  - the **2D quantum gravity** theory
  - dual to 4D $\mathcal{N} = 2$ SUSY through **AGT duality**

We make a different connection through ODE/IM!

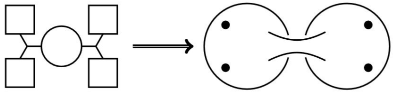


Figure: AGT duality quiver (NLab)

Other ODE/IM examples: Perturbed Hairpin, Paperclip

- **Perturbed Hairpin IM** (for $n = 1, 2$) corresponds to Doubly Confluent Heun eq.

$$-\frac{d^2}{dx^2}\psi(x) + \left[2kqe^x + k^2(e^{2x} + e^{-nx}) + P^2\right]\psi(x) = 0\,. \tag{26}$$

- **Paperclip IM** (for $n = 2$) corresponds to the Confluent Heun eq.

$$-\frac{d^2}{dx^2}\psi(x) + \left[k^2\left(e^x + 1\right)^n - \frac{p^2 e^x}{e^x + 1} - \frac{\left(q^2 - \frac{1}{4}\right)e^x}{\left(e^x + 1\right)^2}\right]\psi(x) = 0\,. \tag{27}$$

- They are both related **2D IQFTs** with **non-conformal boundary interaction**. [Fateev-Lukyanov, hep-th/0510271; Lukyanov-Vitchev-Zamolodchikov, hep-th/0312168]

Fig. 2. The paperclip formed by a junction of two hairpins.

On the origin of ODE/IM

- For a long time, ODE/IM has been regarded as either **mysterious correspondence** or just a mathematical coincidence.
- However, recent work [Fioravanti-Rossi, arXiv:2106.07600] has shed light on its **origin**, by "reversing it":

$QQ/TQ/$Bethe Ansatz

$\Downarrow$

Marchenko-like int. eq.

$\Downarrow$

Schrödinger ODEs

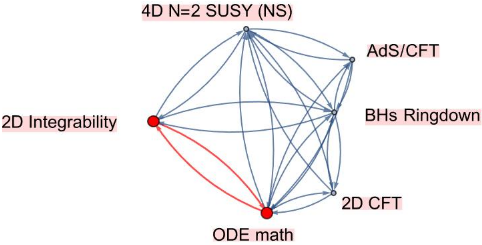


Figure: The Thinker (Rodin)

Three different physical theories, same mathematics!

$$-\frac{d^2}{dy^2}\psi(y) + \left[2e^{2\theta}\cosh y + P^2\right]\psi(y) = 0 \quad \text{INTEGRABILITY (sd-Liouville)} \tag{28}$$

$$\frac{\hbar}{\Lambda_0} = \frac{\epsilon_1}{\Lambda_0} = e^{-\theta} \qquad \frac{u}{\Lambda_0^2} = \frac{1}{2}P^2 e^{-2\theta} \qquad \Downarrow$$

$$-\frac{\hbar^2}{2}\frac{d^2}{dy^2}\psi(y) + [\Lambda^2\cosh y + u]\psi(y) = 0 \quad \text{N=2 GAUGE TH. } (SU(2)\ N_f = 0) \tag{29}$$

$$r = Le^{y/2} \quad \omega L = -2ie^{\theta} \quad P = \frac{1}{2}(l+2) \qquad \Downarrow \qquad \phi(r) = e^{y/2}\psi(y)$$

$$\frac{d^2\phi}{dr^2} + \left[\omega^2\left(1 + \frac{L^4}{r^4}\right) - \frac{(l+2)^2 - \frac{1}{4}}{r^2}\right]\phi(r) = 0 \quad \text{BLACK HOLES PERT. (D3 brane)} \tag{30}$$

- We have shown this **triality** generally for Mathieu, Doubly Confluent Heun, Confluent Heun ODEs.

A more radical method

QNMs as Bethe roots

Black holes perturbation theory

- For perturbations of the **Schwarzschild BH** metric $g^{(0)}_{\mu\nu}$, **Einstein equations**, after **linearization** and **gauge fixing** reduce to just a couple of much simpler PDEs:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{31}$$

$$\Big\Downarrow \quad h_{\mu\nu} = g_{\mu\nu} - g^{(0)}_{\mu\nu} \Rightarrow (Z_l, Q_l)$$

1) for the **axial** perturbations (phase $(-1)^{l+1}$), the **Regge-Wheeler** equation

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right)\right] Q_l(t,x) = 0\,; \tag{32}$$

2) for the **polar** perturbations (phase $(-1)^l$), the **Zerilli** equation

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \left(1 - \frac{2M}{r}\right)\left(\frac{72M^3}{r^5\lambda(r)^2} + (\dots)\frac{12M}{r^3\lambda(r)^2}\left(1 - \frac{3M}{r}\right) + (\dots)\frac{1}{r^2\lambda(r)}\right)\right] Z_l(t,x) = 0, \tag{33}$$

where $\lambda(r) = l(l+1) - 2 + 6M/r$ and $r = r(x)$, with x tortoise coordinate.

From PDEs to ODEs

- The **Laplace transform** $\hat{\Psi}$ of $\Phi = Z_l \vee Q_l$ satisfies the non-homogeneous ODE

$$\hat{\Psi}(s,x) = \int_0^\infty e^{-st}\Phi(t,x)\,dt\,, \qquad \left\{-\frac{\partial^2}{\partial x^2} + U(x) + s^2\right\}\hat{\Psi}(s,x) = -\mathcal{I}(s,x)\,, \tag{34}$$

where $\mathcal{I}(s,x) = -s\Psi(t,x)|_{t=0} - \partial_t\Psi(t,x)|_{t=0}$. The corresponding **homogeneous** equation is the **ODE we will study**:

$$\left\{-\frac{\partial^2}{\partial x^2} + U(x) + s^2\right\}\Psi(s,x) = 0\,. \tag{35}$$

- We consider **solutions** of (35) **regular** for $x \to \pm\infty$, that is infinity and horizon (with $\mathrm{Re}\, s > 0$)

$$\Psi_+(s,x) \sim e^{-sx}\,, \qquad x \to +\infty\,, \qquad \Psi_-(s,x) \sim e^{sx}\,, \qquad x \to -\infty\,. \tag{36}$$

- The Laplace tr. $\hat{\Psi}$ is given by the **Green function** G as (with $x_< = \min(x',x)$, $x_< = \max(x',x)$)

$$\hat{\Psi}(s,x) = \int_{-\infty}^{\infty} G(s,x,x')\mathcal{I}(s,x')dx'\,, \qquad G(s,x,x') = \frac{1}{W[\Psi_-,\Psi_+]}\Psi_-(s,x_<)\Psi_+(s,x_>)\,. \tag{37}$$

Mathematical definition of QNMs

- Finally taking the anti-Laplace transform of $\hat{\Psi}$ and setting $s = i\omega$ we get the **original perturbation** $\Phi = Z_l \vee Q_l$ as [Nollert, Class.Quant.Grav. 16 (1999)]

$$\Phi(t,x) = \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} e^{st}\hat{\Psi}(s,x)\, ds \tag{38}$$

$$= \sum_n e^{i\omega_n t}\mathrm{Res}\left(\frac{1}{W(s)}\right)\Bigg|_{\omega_n} \int_{-\infty}^{\infty}\Psi_-(\omega_n, x_<)\Psi_+(\omega_n, x_>)\mathcal{I}(\omega_n, x')\, dx' + (\text{other contributions})\,.$$

- The **quasinormal modes (QNMs)** are the complex frequencies (real frequencies and damping times) ω_n, which corresponds also to **zeros of the ODE Wronskian**

$$W(\omega_n) \equiv W[\Psi_-, \Psi_+](\omega_n) = 0\,, \tag{39}$$

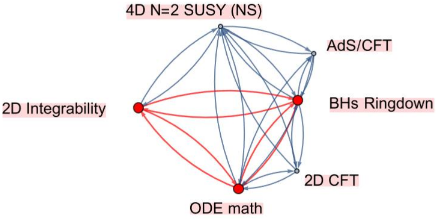
with $W[\Psi_-, \Psi_+] = \Psi_-(x)\Psi'_+(x) - \Psi'_-(x)\Psi_+(x)$.

Quasinormal modes of black holes as Bethe roots

- Through the ODE/IM formula $Q = W[\psi_+, \psi_-]$ we proved that the **Bethe root** (zero) condition on the Baxter's Q function

$$Q(\theta_n) = 0\,, \tag{40}$$

is equivalent to the mathematically rigorous definition of **quasinormal modes (QNMs)**! [Fioravanti-DG, arXiv:2112.11434]



Q has **QNMs as zeroes** and satisfies a **quantum Wronskian functional relation** which can be derived for **nearly any ODE**!

- In general the QQ-system **cannot** be directly inverted and solved as **integral equation**, rather it is necessary to construct **from Q the Y function** that solves the Y **system**.

Y function and quasinormal modes

- For example, for the **triad** [generalized Perturbed Hairpin/ $SU(2)$ $N_f = 2$ NS gauge th./ generalized extremal RN BHs] - *cf.* (3), (26) - one can define a Y **function** as [Fioravanti-DG, arXiv:2112.11434; Fioravanti-DG-Shu, arXiv:2208.14031]

$$Y_{+,+}(\theta) = e^{i\pi(q_1-q_2)}Q_{+,+}(\theta)Q_{-,-}(\theta)\ . \tag{41}$$

- In terms of Y the QQ **system** is written as

$$e^{i\pi(q_1+q_2)}Q_{+,+}(\theta+\frac{i\pi}{2})Q_{-,-}(\theta-\frac{i\pi}{2}) = 1 + Y_{+,-}(\theta)\,. \tag{42}$$

and for **QNMs** follow the alternative **quantizations conditions**

$$Y_{+,-}(\theta_n - i\pi/2) = -1 \qquad Y_{-,+}(\theta_n + i\pi/2) = -1\,. \tag{43}$$

- From the QQ system it follows the Y **system** as

$$Y_{+,-}(\theta+\frac{i\pi}{2})Y_{-,+}(\theta-\frac{i\pi}{2}) = [1+Y_{+,+}(\theta)][1+Y_{-,-}(\theta)]\,. \tag{44}$$

Thermodynamic Bethe Ansatz

- Defining as the **pseudoenergy**

$$\varepsilon_{\pm,\pm}(\theta) = -\ln Y_{\pm,\pm}(\theta)\,, \tag{45}$$

- the Y **system can be inverted in the Thermodynamic Bethe Ansatz (TBA)** for it

$$\begin{aligned}\varepsilon_{\pm,\pm}(\theta) &= \frac{8\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2}e^{\theta} \mp i\pi(q_1 - q_2) - \int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\frac{\ln[1+\exp(-\varepsilon_{\pm,\mp}(\theta'))] + \ln[1+\exp(-\varepsilon_{\mp,\pm}(\theta'))]}{\cosh(\theta-\theta')}\\ \varepsilon_{\pm,\mp}(\theta) &= \frac{8\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2}e^{\theta} \mp i\pi(q_1 + q_2) - \int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\frac{\ln[1+\exp(-\varepsilon_{\pm,\pm}(\theta'))] + \ln[1+\exp(-\varepsilon_{\mp,\mp}(\theta'))]}{\cosh(\theta-\theta')}\,-\end{aligned} \tag{46}$$

- **Technicality**: P enters as the $\theta \to -\infty$ boundary condition $\varepsilon_{+,+}(\theta, P) \sim 4P\theta + 2C(P, q_1, q_2)$, as $\theta \to -\infty$ (following from the perturbative expansion of ODE).

Gravity Y system and TBA

- In **gravity variables** Σ_k (combinations of four charges for **generalised RN extremal BHs**) the Y system reads
[Fioravanti-DG, arXiv:2112.11434; Fioravanti-DG-Shu, arXiv:2208.14031]

$$\begin{aligned} & Y(\theta+\frac{i\pi}{2},-i\Sigma_1,-\Sigma_2,i\Sigma_3,\Sigma_4)Y(\theta-\frac{i\pi}{2},-i\Sigma_1,-\Sigma_2,i\Sigma_3,\Sigma_4) \\ & =[1+Y(\theta,\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4)][1+Y(\theta,-\Sigma_1,\Sigma_2,-\Sigma_3,\Sigma_4)] \end{aligned} \tag{47}$$

- It can be inverted into the **TBA**

$$\begin{aligned} \varepsilon_\pm(\theta) &= [f_{0,+}\mp\frac{i\pi}{2}(\frac{\Sigma_1}{\Sigma_4^{1/4}}-\frac{\Sigma_3}{\Sigma_4^{3/4}})]e^\theta-\varphi*(\bar{L}_\pm+\bar{L}_\mp)(\theta) \\ \bar{\varepsilon}_\pm(\theta) &= [\bar{f}_{0,+}\pm\frac{\pi}{2}(\frac{\Sigma_1}{\Sigma_4^{1/4}}+\frac{\Sigma_3}{\Sigma_4^{3/4}})]e^\theta-\varphi*(L_\pm+L_\mp)(\theta) \end{aligned} \tag{48}$$

where we defined $\varepsilon_\pm(\theta)=-\ln Y(\theta,\pm\Sigma_1,\Sigma_2,\pm\Sigma_3,\Sigma_4)$, $\bar{\varepsilon}_\pm(\theta)=-\ln Y(\theta,\pm i\Sigma_1,-\Sigma_2,\mp i\Sigma_3,\Sigma_4)$, $L=\ln[1+\exp\{-\varepsilon\}]$, $\varphi(\theta)=(2\pi\cosh(\theta))^{-1}$...

Quasinormal modes from TBA

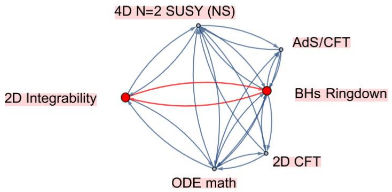
- The **QNMs condition** in gravity variables **reads alternatively as**

$$\bar{Y}_+(\theta_{n'} - i\pi/2) = -1\,, \quad Q_{+,+}(\theta_n) = 0\,,$$
$$\bar{\varepsilon}_+(\theta_{n'} - i\pi/2) = -i\pi(2n'+1)\,. \tag{49}$$

- Through the quantization condition on $\bar{\varepsilon}_+$ we can **actually numerically compute QNMs from TBA**.

n	l	TBA	Leaver	WKB
0	1	$0.86962 - 0.37202i$	$0.86893 - 0.37286i$	$0.896 - 0.3660i$
0	2	$1.47799 - 0.36814i$	$1.47789 - 0.36824i$	$1.494 - 0.3660i$
0	3	$2.08017 - 0.36708i$	$2.08017 - 0.36709i$	$2.091 - 0.3660i$
0	4	$2.68035 - 0.36664i$	$2.68035 - 0.36664i$	$2.689 - 0.3660i$
0	5	$3.27959 - 0.36641i$	$3.27959 - 0.36641i$	$3.287 - 0.3660i$
0	6	$3.87833 - 0.36629i$	$3.87833 - 0.36629i$	$3.884 - 0.3660i$
0	7	$4.47677 - 0.36620i$	$4.47677 - 0.36620i$	$4.482 - 0.3660i$
0	8	$5.07501 - 0.36615i$	$5.07501 - 0.36615i$	$5.080 - 0.3660i$
0	9	$5.67312 - 0.36611i$	$5.67312 - 0.36611i$	$5.677 - 0.3660i$
0	10	$6.27114 - 0.36609i$	$6.27114 - 0.36609i$	$6.275 - 0.3660i$

TABLE II. Comparison of QNMs of (23) with $\Sigma_1 = \Sigma_3 = 0.2$, $\Sigma_2 = 0.4$, $\Sigma_4 = 1$, as obtained from TBA (33) (through (34) with $n' = 0$), Leaver method and WKB approximation.



Comparison of QNMs computations methods

- Computing QNMs exactly is a non-trivial mathematical problem, it has been typically quite **laborious**, also because of their **few exact analytic characterizations**.
- The standard analytic method is the one with the **continued fractions** by Leaver is laborious and sometimes **sometimes not applicable** (in its original form).
- **Physics Informed Neural Networks (PINNs)** can approximate decently complicated cases, but they bring **little new "human understanding"** of black hole physics / ODEs.
- Application of $\mathcal{N}=2$ **gauge theory** is a new analytic characterization, but requires a **nontrivial re-summation procedure** for $\omega_n \sim \Lambda_n \sim 1$.

n	l	TBA	WKB
0	1	$0.89668 - 0.40069i$	$0.931 - 0.3946i$
0	2	$1.53080 - 0.39676i$	$1.551 - 0.3946i$
0	4	$2.78076 - 0.39525i$	$2.792 - 0.3946i$
0	8	$5.26781 - 0.39477i$	$5.274 - 0.3946i$
0	16	$10.2380 - 0.39472i$	$10.238 - 0.3946i$
0	32	$20.1738 - 0.39473i$	$20.165 - 0.3946i$
0	64	$40.0373 - 0.39473i$	$40.019 - 0.3946i$
0	128	$79.7581 - 0.39473i$	$79.729 - 0.3946i$
0	256	$159.185 - 0.39473i$	$159.147 - 0.3946i$

TABLE III. Comparison of QNMs of (23) with $\Sigma_1 = 0.1$, $\Sigma_2 = 0.2$, $\Sigma_3 = 0.3$, $\Sigma_4 = 1$, as obtained from TBA (33), (through (34) with $n' = 0$) and WKB approximation.

Our **integrability** method is **direct and exact**, but now we have developed it for just a **few models**.

A more radical method

Solving QQ-system of gauge theories

Solving QQ systems: D3 brane

- The scalar perturbation on the **D3 brane**, obeys **Mathieu equation** - (25) for $b = 1$, slide 37 - a particular case of DCHE (3), which is the same as for self-dual Liouville IM and $N_f = 0$ quantum SW curve.
- Through linear connection relations among solutions ψ_k one easily derives QQ-systems like

$$Q(\theta + i\pi/2, a)Q(\theta - i\pi/2, a) - Q(\theta, a)^2 = 1\,. \tag{50}$$

- From the QQ **system** follows a **closed formula for** Q

$$Q(\theta, a) = \frac{\sinh\left[\frac{1}{\hbar}A_D(\theta, a)\right]}{\sin\left[\frac{2\pi}{\hbar}a\right]}\,, \quad \text{with} \quad A_D\left(\theta + i\pi/2, a\right) = A_D(\theta, a) - 2\pi i a\,, \tag{51}$$

We proved this and the following formulae purely from **ODE/IM and gauge theory**, but **alternative derivation** is possible through **AGT duality**

[Fioravanti-DG-Mahanta-Rossi,arXiv:24**.*****]

Solving QQ systems: extremal black holes

- We proved the QQ system for $N_f = 2$ gauge theory - *cf.* ODE (3), (26) - that is **extremal BHs**

$$Q\left(\theta + i\pi/2, -m_1, m_2\right) Q\left(\theta - i\pi/2, m_1, -m_2\right) - Q(\theta, -m_1, -m_2) Q(\theta, m_1, m_2) = e^{\frac{i\pi}{\hbar}(m_1 - m_2)} \tag{52}$$

is **equivalent to the shift relations**

$$A_D\left(\theta \pm i\pi/2, m_1, -m_2, a\right) = A_D\left(\theta, m_1, m_2, a\right) + h\left(m_2, a\right) \mp i\pi a \tag{53}$$

$$(-\partial_{m_1} - \partial_{m_2})\mathcal{F}_{inst}(\theta + i\pi/2, -m_1, -m_2, a) + (\partial_{m_1} + \partial_{m_2})\mathcal{F}_{inst}(\theta - i\pi/2, m_1, m_2, a) = 0$$

and similar others, with $h(m, a) = \ln \sqrt{\cos \frac{\pi}{\hbar}(a - m) \sec \frac{\pi}{\hbar}(a + m)}$.

- The **exact closed formula** for Q for $N_f = 2$ ensues

$$Q(\theta, m_1, m_2, a) = 2\pi \exp\left\{\frac{1}{\hbar}\sum_{k=1}^{2} \frac{\partial \mathcal{F}\left(\theta, m_1, m_2, a\right)}{\partial m_k}\right\} \frac{\sinh \frac{1}{\hbar} A_D\left(\theta, m_1, m_2, a\right)}{\sin \frac{2\pi}{\hbar} a}. \tag{54}$$

Solving QQ systems: general (Minkowski) black holes (I)

- We proved the first QQ system for $N_f = 3$ gauge theory - *cf.* ODE (5), (2), (27) - that is **Schwarschild BHs**

$$Q_1(\theta+i\pi, m_1, m_2, -m_3)Q_1(\theta, -m_1, -m_2, m_3) - Q_1(\theta, m_1, m_2, m_3)Q_1(\theta+i\pi, -m_1, -m_2, -m_3) = \frac{2i}{\hbar}(m_1+m_2)\,, \tag{55}$$

  is **equivalent to the shift relations**

$$A_D\left(\theta \pm i\pi, m_1, m_2, -m_3, a\right) = A_D\left(\theta, m_1, m_2, m_3, a\right) + h\left(m_3, a\right) \mp i\pi a\,, \tag{56}$$

$$(\partial_{m_1}+\partial_{m_2}-\partial_{m_3})\mathcal{F}_{inst}(\theta+i\pi, m_1, m_2, -m_3, a) + (-\partial_{m_1}-\partial_{m_2}+\partial_{m_3})\mathcal{F}_{inst}(\theta, -m_1, -m_2, m_3, a) = 0$$

  and similar others.

- The **exact closed formula** for Q_1 ensues

$$Q_1\left(\theta, \mathbf{m}, a\right) = \tag{57}$$

$$\pi i 2^{\frac{3}{2}} e^{\frac{1}{\hbar}\left[m_3\theta + \left(\frac{m_1+m_2}{2}+m_3-\frac{\Lambda_3}{16}\right)\ln 2\right]}\Gamma\left(1+\frac{m_1+m_2}{\hbar}\right)\exp\left\{\frac{1}{\hbar}\sum_{k=1}^{3}\frac{\partial\mathcal{F}\left(\theta, \mathbf{m}, a\right)}{\partial m_k}\right\}\frac{\sinh\frac{1}{\hbar}A_D\left(\theta, \mathbf{m}, a\right)}{\sin\frac{2\pi}{\hbar}a}\,.$$

Solving QQ systems: general (Minkowski) black holes (II)

- We proved the second QQ system for $N_f = 3$ gauge theory, that is **Schwarschild BHs**

$$Q_2(\theta, -q_2, -q_1, q_3) Q_2(\theta, q_2, q_1, q_3) - Q_2(\theta, q_1, q_2, q_3) Q_2(\theta, -q_1, -q_2, q_3) = q_1^2 - q_2^2 \tag{58}$$

is equivalent to the shift relation

$$\partial_{m_1} \mathcal{F}_{inst}(\theta, m_1, m_2, m_3, a) - \partial_{m_1} \mathcal{F}_{inst}(\theta, -m_1, -m_2, m_3, a) = 0\,. \tag{59}$$

and similar others

- The **exact closed formula** for Q_2 ensues

$$Q_2\left(\theta, \mathbf{m}, a\right) = 2^{\frac{m_2}{\hbar}} \Gamma(1 + \frac{m_1 + m_2}{\hbar}) \Gamma(1 + \frac{m_1 - m_2}{\hbar}) \exp\left\{ \frac{2}{\hbar} \frac{\partial \mathcal{F}\left(\theta, \mathbf{m}, a\right)}{\partial m_1} \right\}\,. \tag{60}$$

[Fioravanti-DG-Mahanta-Rossi,arXiv:24**.*****]

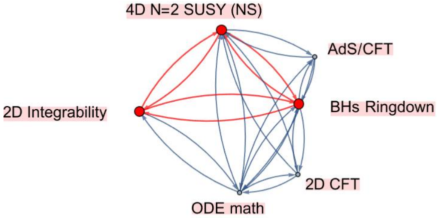
Proof of new ($\mathcal{N} = 2$) gauge - gravity duality

- Since proved a relation between **at least one** Q **function and the gauge dual period**

$$Q(\theta, \mathbf{m}, a) = \mathfrak{F}(\theta, \mathbf{m}, a) \frac{\sinh \frac{1}{\hbar} A_D(\theta, \mathbf{m}, a)}{\sin \frac{2\pi}{\hbar} a}, \qquad (61)$$

from our formalism **it follows immediately that QNMs are given by quantization conditions on the gauge periods** *cf.* (11)

$$Q(\theta, \mathbf{m}, a) = 0 \quad \Longrightarrow \quad A_D(\theta, \mathbf{m}, a) = 2i\pi\hbar\, n\,. \qquad (62)$$



- (62) was previously found heuristically and allowed:
  - a **novel analytic characterization of QNMs**;
  - to **find new results for the BHs theory** (by many other authors);
  - an **unexpected application of Supersymmetry**.

Outlook

Conclusions and future developments

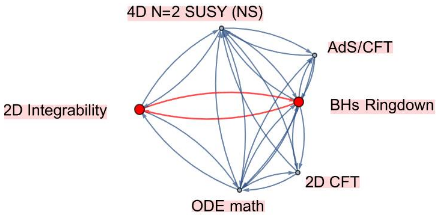
Application scope of integrability

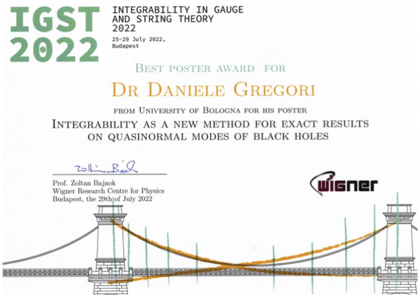
We notice that **much of the BH theory seems to go in parallel to the ODE/IM correspondence**, beyond the determination of QNMs!

- For instance, also the **greybody factor** that parametrizes the **Hawking radiation** seems to be **ratio of Qs**
- Intepretation and use of T, TQ relation...
  [Fioravanti-DG, arXiv:2112.11434; Fioravanti-DG-Shu, arXiv:2208.14031; work in progress...]

Future **theoretical problems** to be addressed:

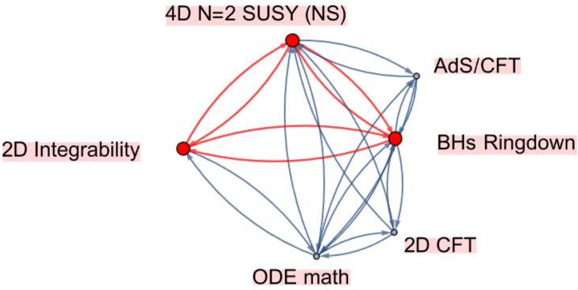
- Understanding **overtones instability**?
- Understanding **vanishing of Love numbers**?





Summary

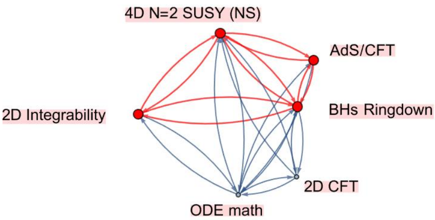
- We have shown **BHs perturbation theory** has a natural connection with
  - 4D $\mathcal{N} = 2$ **NS SUSY gauge theory**;
  - 2D **integrable models**;
  - many other related fields.
- In particular **ODE/IM correspondence** allowed
  - **understand the relation** among the three;
  - **find new results** an all sides, at the **exact level**.

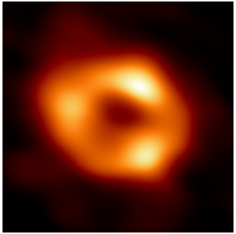


- Especially for QNMs of BHs we give:
  - **new analytic characterizations** from Q, Y, T functions;
  - **new method of computation** as Thermodynamic Bethe Ansatz.

Future developments

- $\mathcal{N} = 2$ gauge theories and BHs constitute **novel fields of application of integrability** and many future developments could be pursued:
  - generalization to **more BHs models**;
  - extension to **other BHs observables**;
  - addressing **theoretical problems in BH physics**;
  - tackling **non-linear** perturbation theory;
  - **connection to other fields** (like holography).





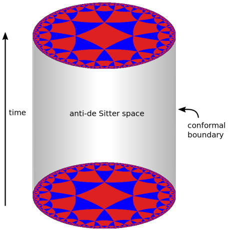
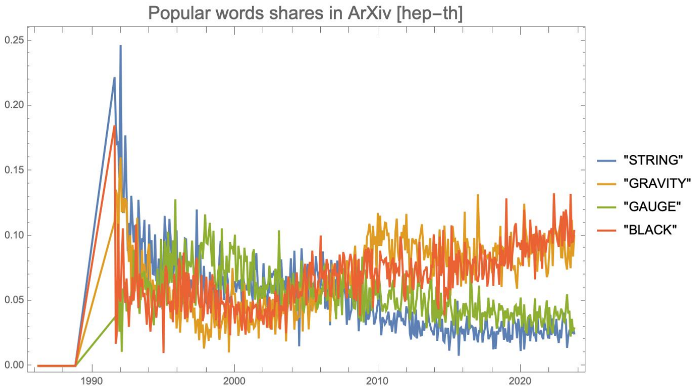


Figure: (Wikicommons/Herzog,Dunkel)

Thank you for your attention!



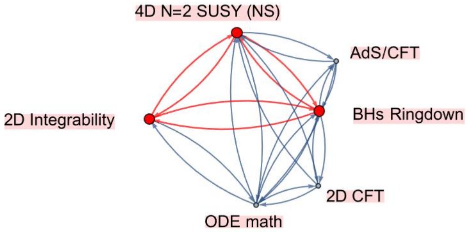


Figure: - GitHub

Figure: - Inspire