

Integrability and cycles of deformed $\mathcal{N} = 2$ gauge theory

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LIJC Gong Show

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Based on:

[arXiv:1908.08030](#), [arXiv:1909.11100](#) and [to appear soon](#)
with D. Fioravanti, H. & R. Poghossian and H. Shu.

Outline

- 1 Introduction to ODE/IM
- 2 Introduction to deformed Seiberg-Witten theory
- 3 Integrability-Gauge connection
 - Proof of the fundamental identifications
 - Gauge interpretation of integrability functional relations
 - Gauge interpretation of local integrals of motion
- 4 Generalisations
- 5 Conclusion and perspectives

Introduction to ODE/IM

- In the ODE/IM (Ordinary Differential Equations / Integrable Models) Correspondence, one can compute and derive all integrability functions (Baxter's Q , Y , T ...) and functional relations starting from the solutions of some ODE related to some IM. [Dorey-Tateo:1999; Lukyanov-Bazhanov-Zamolodchikov:1999; Gaiotto-Moore-Neitzke:2010]
- It was discovered by Alexei Zamolodchikov [Zamolodchikov:2012] that the following ODE (Generalized Mathieu Equation, GME)

$$\left\{ -\frac{d^2}{dy^2} + e^{2\theta}(e^{y/b} + e^{-yb}) + P^2 \right\} \psi(y) = 0, \quad (1)$$

is associated to the Liouville integrable model (for L. coupling b , momentum P and rapidity θ) [Zamolodchikov:2006].

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Introduction to deformed Seiberg-Witten theory

- In 4D $\mathcal{N} = 2$ SYM theories, to include instanton contributions, it is necessary to deform spacetime through two parameters, ϵ_1 and ϵ_2 (Ω -background). [Nekrasov:2004, Nekrasov-Okounkov:2006, Nekrasov:2009]
 - When both $\epsilon_1, \epsilon_2 \rightarrow 0$ we have the Seiberg-Witten (SW) original theory.
 - An intermediate limit which we will study is the Nekrasov-Shatashvili (NS): $\epsilon_1 = \hbar \neq 0$, $\epsilon_2 \rightarrow 0$ [Nekrasov-Shatashvili:2009].
- The NS deformed/quantized SW differential is obtained from the solution of the Mathieu equation

$$-\frac{\hbar^2}{2} \frac{d^2}{dz^2} \psi(z) + [\Lambda^2 \cos z - u] \psi(z) = 0. \quad (2)$$

where u parametrizes the moduli space of vacua and Λ is a scaling parameter.
[Alday-Gaiotto-Tachikawa:2010; Gaiotto:2013; Awata-Yamada:2010]

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- The **deformed prepotential** $\mathcal{F}_{\text{NS}}$ (logarithm of the partition function) may be derived by eliminating u between the **two deformed cycles (periods)**

$$a(\hbar, u, \Lambda) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{P}(z; \hbar, u, \Lambda) dz, \quad a_D(\hbar, u, \Lambda) = \frac{1}{2\pi} \int_{-\arccos(u/\Lambda^2) - i0}^{\arccos(u/\Lambda^2) - i0} \mathcal{P}(z; \hbar, u, \Lambda) dz \quad (3)$$

integrals of the quantum SW differential $\mathcal{P}(z) = -i \frac{d}{dz} \ln \psi(z)$.

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Integrability-Gauge fundamental correspondence

- The self-dual ($b = 1$) GME is known in literature as **modified Mathieu equation**:

$$\left\{ -\frac{d^2}{dy^2} + 2e^{2\theta} \cosh y + P^2 \right\} \psi(y) = 0, \quad (4)$$

- This equation can be related to the Mathieu equation which quantizes the SW differential in the NS limit (2) by the change of var. $z = -iy - \pi$ and the **parameters correspondence**

$$\frac{\hbar}{\Lambda} = \frac{\epsilon_1}{\Lambda} = e^{-\theta} \quad \frac{u}{\Lambda^2} = \frac{1}{2} \frac{P^2}{e^{2\theta}}. \quad (5)$$

- We (**D. Fioravanti and D. Gregori - arXiv:1908.08030**) have found that the relations:

$$Q(\theta, P^2) = \exp 2\pi i a_D(\epsilon_1, u), \quad (6)$$

$$T(\theta, P^2) = 2 \cos 2\pi a(\epsilon_1, u). \quad (7)$$

between the two NS deformed Seiberg-Witten cycle periods a, a_D for **pure** ($N_f = 0$) $SU(2) \mathcal{N} = 2$ SYM and the Baxter's Q and T functions of the **self-dual Liouville integrable model**.

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Proof of the fundamental identification for Q

- We have proven relation (6) analytically by studying the properties of the solution $\mathcal{P}(y) = -i \frac{d}{dy} \ln \psi(y)$ of the Riccati equation, equivalent to the ODE (4) with boundary condition given by the (double) limit $y \rightarrow +\infty$ of the SW leading $\hbar \rightarrow 0$ order. [Fioravanti,Gregori:2019]

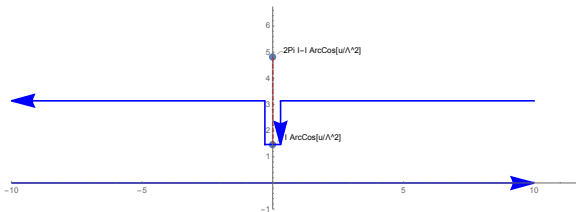


Figure: Integration contour in the y complex plane for the proof.

$$\underbrace{\int_{-\infty}^{+\infty} \mathcal{P}_{reg}(y) dy}_{\ln Q(\theta, P^2)} = \underbrace{\int_{-\arccos(u/\Lambda^2) - i0}^{+\arccos(u/\Lambda^2) - i0} \mathcal{P}(z) dz}_{2\pi i a_D(\hbar, u)} \quad (8)$$

$$Q(\theta, P^2) = \exp 2\pi i a_D(\epsilon_1, u), \quad (9)$$

Proof of the fundamental identification for T

- Alexei Zamolodchikov proved numerically a relation between the $b = 1$ T and the Floquet index ν of the Mathieu equation, such that $\psi(z + 2\pi) = e^{2\pi\nu}\psi(z)$. [Zamolodchikov:2012]

$$T = 2 \cosh 2\pi\nu. \quad (10)$$

- He did not recognise that $\nu = ia$ (NS limit of $\mathcal{N} = 2$ SYM not yet existing), but we did it, thus establishing another connection to gauge theory [Fioravanti,Gregori:2019]

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Gauge TBA and R -symmetry

- The self-dual Liouville **QQ relation** for $Q(\theta, u) \equiv Q(\theta, P^2)$ translates in the gauge variables as

$$1 + Q^2(\theta, u) = Q(\theta - i\pi/2, -u)Q(\theta + i\pi/2, -u) \quad (12)$$

and can be inverted to obtain a **TBA for the dual period** ($e^{-\varepsilon(\theta, u)} = Q^2(\theta, u)$)

$$\varepsilon(\theta, u) = -4\pi i a_D^{(0)}(u) \frac{e^\theta}{\Lambda} - 2 \int_{-\infty}^{\infty} \frac{\ln [1 + \exp\{-\varepsilon(\theta', -u)\}]}{\cosh(\theta - \theta')} \frac{d\theta'}{2\pi} \quad (13)$$

- The TBA enjoys the $\mathbb{Z}_2$ **R -symmetry** of the moduli space $u \leftrightarrow -u$, which is nothing but the **discrete symmetry** used for the **ODE/IM construction**.

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Non perturbative $\mathbb{Z}_2$ symmetry relations

- The self-dual Liouville TQ relation and T periodicity relation read in gauge variables

$$T(\theta, u) = \frac{Q(\theta - i\pi/2, -u) + Q(\theta + i\pi/2, -u)}{Q(\theta, u)}, \quad (14)$$

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- These relations appear to be a non-perturbative exact generalization of the perturbative ($\theta \rightarrow +\infty, \hbar \rightarrow 0$) $\mathbb{Z}_2$ R -symmetry relations for the periods [Bilal-Ferrari:1996, Basar-Dunne:2015]

$$a_D^{(n)}(-u) = i(-1)^n \left[-\operatorname{sgn}(\operatorname{Im} u) a_D^{(n)}(u) + a^{(n)}(u) \right], \quad (16)$$

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Perturbative periods and local integrals of motion

- By taking a double limit $\theta, P^2 \rightarrow +\infty$ on the integrability asymptotic expansion of Q , the LIMs l_{2n-1} resum to the perturbative periods $a_D^{(n)}$.

$$2\pi i a_D^{(n)}(u, \Lambda) = -\Lambda^{1-2n} \sum_{k=0}^{\infty} 2^k C_{n+k} \Upsilon_{n+k,k} \left(\frac{u}{\Lambda^2} \right)^k. \quad (18)$$

where $l_{2n-1}(b=1, P^2) = \sum_{k=0}^n \Upsilon_{n,k} P^{2k}$.

- We have derived from the series (18) the quantum Picard-Fuchs equations at all perturbative orders (through an algorithm) for $a^{(n)}$ and $a_D^{(n)}$. For instance

$$\left\{ (u^2 - \Lambda^4) \frac{\partial^2}{\partial u^2} + 6u \frac{\frac{u^2}{\Lambda^4} + \frac{111}{8}}{\frac{u^2}{\Lambda^4} + \frac{325}{32}} \frac{\partial}{\partial u} + \frac{21}{4} \frac{\frac{u^2}{\Lambda^4} + \frac{689}{32}}{\frac{u^2}{\Lambda^4} + \frac{325}{32}} \right\} a_D^{(2)}(u, \Lambda) = 0. \quad (19)$$

- Viceversa, we can interpret in integrability this equations as fixing the LIMs for $b=1$.

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Generalisations

- We have described our gauge-integrability connection just for the simplest $SU(2)$ with $N_f = 0$ case, but we have evidence that it is of much more general validity.
 - For the case of $SU(3)$ with $N_f = 0$, D.Fioravanti, R.Poghossian and H.Poghosyan have found a relation between the 3 periods and the T function of the A_2 Toda Integrable model, which is a generalisation of (7). [Fioravanti-Poghossian-Poghosyan:2019]
 - For the case of $SU(2)$ with $N_f = 1$, D. Fioravanti, D.Gregori and H. Shu have found a relation between the 2 periods and the Y of the Integrable Perturbed Hairpin model. [Fioravanti-Gregori-Shu:to appear soon]
- A different kind of connection between Q and Y functions and gauge periods has been found in [Grassi-Gu-Marino:2019, Grassi-Hao-Neitzke:2021]. However, it turns out that they are different, since they involve different periods, generated by the instanton prepotential rather than by the cycle integrals.

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Summary dictionary

- In conclusion, the powerful ODE/IM correspondence has been revealing a **very suggestive connection** between the **2D quantum integrable models** and **4D NS (ϵ_1)-deformed Seiberg-Witten $\mathcal{N} = 2$ supersymmetric gauge theories**.

2D Integrability	4D NS-def. $\mathcal{N} = 2$ SYM
Q, Y function	(dual) period a_D
T function	period a
ODE/IM discrete sym.	$\mathbb{Z}_n$ R-symmetry
Q, Y system	TBA for gauge period
TQ relation	R-symmetry exact rel. for a_D
T periodicity	R-symmetry exact rel. for a
L.I.M. I_{2n-1}	quantum P.F. eq. for $a^{(n)}, a_D^{(n)}$

Conclusions and perspectives

- It would be also interesting to explore:
 - the correspondence with higher $SU(2)$ flavours $N_f = 2, 3, 4$, which is related also to the computation of quasi-normal-modes frequencies of black holes merging; [Aminov-Grassi-Hatsuda:2020, Bianchi-Consoli-Grillo-Morales:2021]
 - the problem of wall-crossing of TBA, that is, the gauge TBA also in the weak coupling regime (should be related to TQ relation);
 - the generalization to the full Omega background, with both $\epsilon_1, \epsilon_2 \neq 0$ (a PDE/IM correspondence would need to be developed);
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THANK YOU!