$\mathcal{N} = 2$ deformed gauge theory and CFT_2 for BHs and GWs observations (Seiberg-Witten / Quasinormal Modes correspondence)

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NORDITA, 22nd September 2021

Review based on:
arXiv:2006.06111, by G. Aminov, A. Grassi, Y. Hatsuda;
arXiv:2007.07906, by Y. Hatsuda;
arXiv:2105.04245, by M. Bianchi, D. Consoli, A. Grillo, J. F. Morales;
arXiv:2105.04483, by G. Bonelli, C. Iossa, D. Panea Lichtig, A. Tanzini;

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Introduction

GW: a probe for new physics

- The recent **experimental verification of gravitational waves** renewed the **interest in the theoretical studies** of General Relativity and black hole physics. [4]
- Direct detection of gravitational waves (GW) allows to **test General Relativity** (GR) in extreme regimes and to **search for possible deviations** from it, that is, new physics. [3]
- In particular, it becomes possible to **discriminate Black Holes (BHs) from fuzzballs or other exotic compact objects (ECOs)** in terms of their multipoles, shadows or tidal effects. [3]

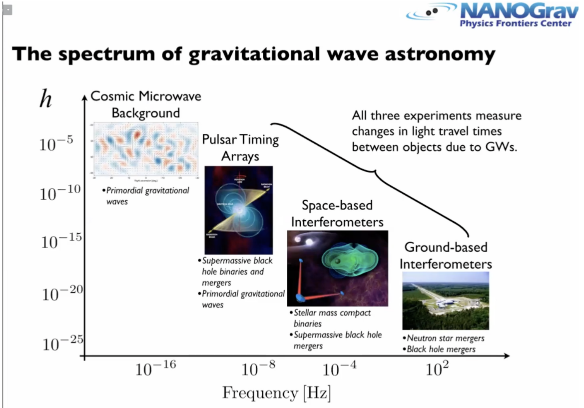

Figure: Credit: Xavier Siemens / Strings 2021

Colliding BH, ringdown phase: QNM frequencies and Spin-weighted spheroidal harmonics

- A **black hole collision** can be divided in 3 phases: **inspiral, merger and ringdown**.
- The **quasinormal modes (QNMs)** are responsible for the **damped oscillations** appearing, for example, in the **ringdown phase** of two colliding BH and have a direct connection to **gravitational waves observations**. [1]
- The QNM spectra are discrete and complex. The QNMs do not correspond to bound states (or normal modes) but rather to **resonance states** (or dissipative modes) in quantum mechanics. [1]
- The (eigenvalue) of the field encoding the angular part of the perturbation (for Kerr BHs) is called **spin-weighted spheroidal harmonics** (see slide (25) below).

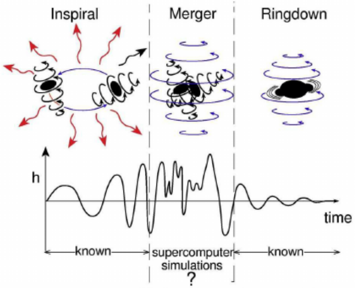

Figure: Credit: Kip Thorne

New physics from QNM

- In the case of **ECOs in alternative theories of gravity** and **fuzzballs in string theory**, the "prompt ring-down signal" decomposes in the early-stage resonant modes, produced around their "photon-spheres" that **may differ from the BH** ones [3], because of **horizon scale structure** [7].
- At later stages both ECOs and fuzzballs produce a peculiar train of **echoes**, probing their internal "cavity" and not only their external "walls", with **significant deviations from GR**. [3]

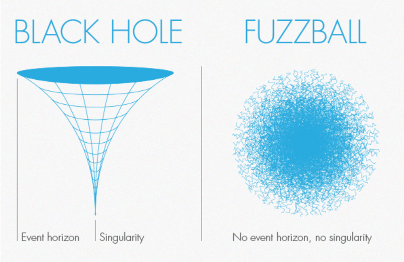

Figure: Credit: Olena Shmahalo/Quanta Magazine

- The **crucial role played by QNMs in discriminating** BHs from fuzzballs or other ECOs motivated renewed effort in their **determination with higher and higher accuracy**. [3]
- One immediate outcome of the analysis of [1, 3, 4] is a **better knowledge of the QNMs** (for the BHs considered).

QNM: numerics and analytics

- The QNMs are **only known numerically**, with high precision, though. [3]
- There are a lot of ways to compute *numerically* QNMs, but **until [1] analytic approaches were less developed**.

Nb	$2M\omega_0(0,0)$
3	**0.21**453301 − **0.20**342058i
8	**0.22**088781 − **0.20**978038i
12	**0.22090951 − 0.2097913**1i
Num	**0.22090988 − 0.20979143i**

Figure: Example of QNM frequencies from [1]

- In the **eikonal approximation**, the real and imaginary parts of the QNM frequencies or the "prompt ring-down modes" can be expressed as

$$\omega_{QNM} \simeq \omega_c(l) - i(2n+1)\lambda_c\,, \tag{1}$$

where $\omega_c(l)$ is the orbital frequency of unstable "circular" orbits forming the light-ring, while λ_c is the Lyapunov exponent governing the chaotic behaviour of nearly critical geodesics around it [3].

The importance of exact solutions of DEs

- To have **high precision tests of General Relativity** equations, exact computational techniques can be used.
- The **study of exact solutions of differential equations** rather than their approximate or numerical solutions is of paramount importance both to deepen our comprehension of physical phenomena and to reveal possible physical fine structure effects. [4]
- Recent developments in the study of **two-dimensional conformal field theories**, their relation with **supersymmetric gauge theories**, equivariant localisation and duality in **quantum field theory** produced **new tools which are very effective** to study long-standing classical problems in the theory of differential equations. [4]
- Actually [3] consider also *difference* (not *differential*) equations for gauge theories (see below slide 44).

General aspects of 4D $\mathcal{N} = 2$ SYM / DD BH correspondence

Classical SW curve and gauge periods

- Given an operator $H(\hat{x},\hat{p})$, with $[\hat{x},\hat{p}]=i\hbar$, we associate it with a **classical curve and a one form**

$$H(x,p)=E\,, \qquad \lambda(x,E)=p(x,E)\,d\,x \tag{2}$$

- We consider operators $H(\hat{x},\hat{p})$ such that the corresponding classical curve (2) coincides with a **SW curve** of a suitable **4D $\mathcal{N}=2$ gauge theory**.
- If the curve (2) has genus g, we choose a basis of cycles $\mathcal{A}_i,\mathcal{B}_i$, with $i=1,\dots,g$ and define classical periods by integrating over such cycles

$$\Pi_{A_i,B_i}(E)=\oint_{\mathcal{A}_i,\mathcal{B}_i} p(x,E)\,d\,x\,, \qquad i=1,\dots,g\,. \tag{3}$$

- (3) are the **SW periods** and they encode the **central charges and masses of the BPS particles** in the theory.

Quantum SW curve and gauge periods from resummation

- **Quantum Seiberg Witten curve** in the **Nekrasov-Shatashvili limit** of the Ω-background (used to compute instanton contributions) $\epsilon_2 \to 0, \epsilon_1 = \hbar$ [1] (see for more details slide 43)

$$H_{N_f}\psi(x) = E\psi(x) \tag{4}$$

$$H_{N_f} = -\hbar^2 \frac{d^2}{dx^2} + \frac{1}{2}\Lambda_{N_f}^{2-N_f/2}\left[e^x K_+\left(\hbar\frac{d}{dx} - u_1 + \frac{\hbar}{2}\right) + e^{-x}K_-\left(\hbar\frac{d}{dx} - u_1 - \frac{\hbar}{2}\right)\right] - u_0 \ . \tag{5}$$

- Define exact periods by **resummation** of their $\hbar \to 0$ asymptotic expansions. [1]

$$\Pi^{\mathrm{WKB}}_{A_i,B_i}(E,\hbar) \doteq \sum_{n\geq 0}\hbar^n \oint_{\mathcal{A}_i,\mathcal{B}_i} Q_n(x,E)\,dx \quad \Longrightarrow \quad \Pi_{A_i,B_i} \tag{6}$$

- In practice, resummation can be done either through **TBA** or through **instanton counting**. [1] chose the latter.
- The need of resummation is **why supersymmetric gauge theory plays a crucial role** in the problem. [1]

Gauge periods from instanton calculus

- The **quantum periods** are encoded by: [1]
  - the **Nekrasov-Shatashvili free energy** $\mathcal{F}^{(N_f)}(a, \mathbf{m}, \Lambda_{N_f}, \hbar)$, which can be computed systematically (at least at the instanton level) through combinatorial calculus on Young Tableaus of the gauge group representation;
  - the **4D quantum mirror map** $a(E, \mathbf{m}, \Lambda_{N_f}, \hbar)$.
- That is

$$\Pi_A^{(N_f)}(E, \mathbf{m}, \Lambda_{N_f}, \hbar) = a(E, \mathbf{m}, \Lambda_{N_f}, \hbar) \tag{7}$$

and

$$\Pi_B^{(N_f)}(E, \mathbf{m}, \Lambda_{N_f}, \hbar) = \partial_a \mathcal{F}^{(N_f)}(E, \mathbf{m}, \Lambda_{N_f}, \hbar)\Big|_{a=a(E,\mathbf{m},\Lambda_{N_f},\hbar)} . \tag{8}$$

- The **Matone's relation**

$$E = a^2 - \frac{\Lambda_{N_f}}{4-N_f} \frac{\partial \mathcal{F}_{\text{inst}}^{(N_f)}(a; \mathbf{m}, \Lambda_{N_f}, \hbar)}{\partial \Lambda_{N_f}} . \tag{9}$$

can be inverted leading to the four dimensional quantum mirror map [1].

Quantization condition, resonances

- The discrete spectrum of H_{N_f} is captured by the following **quantization condition**: [1]

$$\Pi_I^{(N_f)}(E, \mathbf{m}, \Lambda_{N_f}, \hbar) = \mathcal{N}_I \left(n + \frac{1}{2}\right), \qquad I = A, B, \qquad n = 0, 1, 2, \ldots, \tag{10}$$

- On one hand one usually imposes suitable positivity conditions on

$$\hbar\,, \Lambda_{N_f}\,, m_i \tag{11}$$

  so that H_{N_f} has a real, discrete spectrum.

- On the other hand if $\hbar\,, \Lambda_{N_f}\,, m_i$ are **complex**, we can think of the spectral problem in terms of **resonances**. Since BH QNMs are nothing but resonances, **we strongly expect** that their spectra are computed in the geometric/gauge framework. [1]

General recipe

The **recipe** for setting up the correspondence is the following: [1]

1. read off the **singularity structure** of the ODE;
2. look for **gauge theory counterparts** by comparing the Riemann sphere with punctures associated with quiver gauge theories;
3. find relations between the **parameters**.

2D CFT: physical explanation?

- In the view of [4], the **2D CFT** framework, connected to 4D $\mathcal{N}=2$ SYM through **AGT duality**, is the suitable one to provide a **physical explanation of the relation between black hole physics and supersymmetric gauge theories**.
- By using the explicit solution of the connection problem, [4] provide a **proof of the exact quantization of Kerr black hole quasinormal modes** as proposed in [1]. By solving the angular Teukolsky equation, [4] also **prove the analogue dual quantization condition on the corresponding parameters of the spin-weighted spheroidal harmonics** (see section 5 for further explanations).
- [6] argue that the use of the AGT correspondence may also provide **additional information in discriminating BHs from fuzzballs**.

Simple explicit cases

Schwarzschild BH / $SU(2)$ $N_f = 3$ $\mathcal{N} = 2$ SYM - I

- Schwarzschild metric for **static and spherically symmetric solutions** to the Einstein equation in the vacuum

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{12}$$

- Scalar ($s = 0$), electromagnetic ($s = 1$) and odd-parity gravitational ($s = 2$) **linear perturbations of the metric** are governed by the **Regge-Wheeler type equation**

$$\left[f(r)\frac{d}{dr}f(r)\frac{d}{dr} + \omega^2 - V(r)\right]\phi(r) = 0\,, \tag{13}$$

$$f(r) = \left(1 - \frac{2M}{r}\right)\,, \qquad V(r) = f(r)\left[\frac{l(l+1)}{r^2} + (1 - s^2)\frac{2M}{r^3}\right]\,, \tag{14}$$

$$\text{boundary conditions} \qquad \phi(r) \sim \begin{cases} e^{-i\omega(r+2M\ln(r-2M))} & r \to 2M \\ e^{+i\omega(r+2M\ln(r-2M))} & r \to \infty \end{cases} . \tag{15}$$

Schwarzschild BH / $SU(2)$ $N_f = 3$ $\mathcal{N} = 2$ SYM - II

- By the change of variable [1]

$$r = 2Mz\,, \qquad \phi(r) = \sqrt{\frac{z}{z-1}}\Phi(z)\,, \tag{16}$$

(13) becomes the equation

$$\Phi''(z) + \tilde{Q}(z)\Phi(z) = 0 \qquad \tilde{Q}(z) = \frac{1}{z^2(z-1)^2}\sum_{i=0}^{4}\tilde{A}_i z^i \tag{17}$$

$$\tilde{A}_0 = -s^2 + \frac{1}{4}\,, \quad \tilde{A}_1 = l(l+1) + s^2\,, \quad \tilde{A}_2 = -l(l+1)\,, \quad \tilde{A}_3 = 0\,, \quad \tilde{A}_4 = (2M\omega)^2\,. \tag{18}$$

- Equation (17) has two regular singularities at $z = 0, 1$ and one irregular singularity (of Poincaré rank 1) at $z = \infty$. It corresponds to the **Confluent Heun** equation.

Schwarschild BH / $SU(2)$ $N_f = 3$ $\mathcal{N} = 2$ SYM - III

- which stands in correspondence with the $SU(2)$ **quantum Seiberg-Witten curve for** $N_f = 3$ by the parameters correspondence [1]

$$\begin{aligned} \hbar = 1\,, \qquad \Lambda_3 = -16iM\omega\,, \qquad E = -l(l+1) + 8M^2\omega^2 - \frac{1}{4}\,, \\ m_1 = s - 2iM\omega\,, \qquad m_2 = -s - 2iM\omega\,, m_3 = -2iM\omega\,. \end{aligned} \tag{19}$$

- $$\left[-\hbar^2\partial_x^2 + Q_3(x)\right]\tilde{\psi}(x) = 0 \tag{20}$$

$$Q_3(x) = \frac{e^{-2x}}{16(\sqrt{\Lambda_3}e^x - 2)^2}\Big[4\Lambda_3 + 4\Lambda_3 e^{4x}(m_1 - m_2)^2 + \Lambda_3 e^{2x}(\Lambda_3 - 24m_3) \tag{21}$$

$$+ 4\sqrt{\Lambda_3}e^{3x}(8m_1m_2 + \Lambda_3 m_3 - 2\hbar^2) - 4\sqrt{\Lambda_3}e^x(\Lambda_3 - 8m_3)\Big] + \frac{2E}{\sqrt{\Lambda_3}e^x - 2} \tag{22}$$

Schwarzschild BH / $SU(2)$ $N_f = 3$ $\mathcal{N} = 2$ SYM - IV

$$\psi(x) = \exp\left[\frac{e^{-x}\{e^x[\Lambda_3 x - 4(m_1 + m_2 + \hbar)\ln(2 - \sqrt{\Lambda_3}e^x)] - 2\sqrt{\Lambda_3}\}}{8\hbar}\right]\tilde{\psi}(x) \tag{23}$$

$$z = \frac{2e^{-x}}{\sqrt{\Lambda_3}} \quad \tilde{\psi}(x) = z^{-1/2}\Psi(z) \tag{24}$$

$$\hbar^2\Psi''(z) + \frac{1}{z^2(z-1)^2}\sum_{i=0}^{4}\hat{A}_i z^i \Psi(z) = 0 \tag{25}$$

Schwarzschild BH / $SU(2)$ $N_f = 3$ $\mathcal{N} = 2$ SYM - V

- The **quantization condition** for $SU(2)$ $N_f = 3$ $\mathcal{N} = 2$ SYM in terms of Schwarzschild BHs parameters reads [1]

$$\Pi_B^{(3)}\left(-l(l+1) + 8M^2\omega^2 - \frac{1}{4}, \mathbf{m}, -16iM\omega, 1\right) = 2\pi\left(n + \frac{1}{2}\right), \qquad n = 0, 1, 2, \ldots, \tag{26}$$

$$\mathbf{m} = \{s - 2iM\omega, -s - 2iM\omega, -2iM\omega\}. \tag{27}$$

- For a given set of quantum numbers $\{l, s, n\}$, this equation admits a **discrete family of complex solutions** $\omega_n(l, s)$ which give the **QNMs** $M\omega_n$. [1]

- 1. The LHS of (26) is computed by Nekrasov's combinatorial formula systematically.
  2. The problem is that the NS free energy includes the parameter a that is not directly related to the black hole parameters. To avoid it we use Matone relation (9).
  3. Therefore we can finally eliminate a from the NS free energy, and thus we can solve the quantization condition (26) with respect to $M\omega$.

Kerr BH / $SU(2)$ $N_f = 3$ $\mathcal{N} = 2$ SYM - I

- Kerr metric for **stationary and axially symmetric solutions** to the Einstein equation in the vacuum (in Boyer-Lindquist coordinates, with mass M, angular momentum α)

$$ds^2 = -dt^2 + dr^2 + 2\alpha \sin^2\theta dr d\phi + (r^2 + \alpha^2 \cos^2\theta) d\theta^2 + (r^2 + \alpha^2)\sin^2\theta d\phi^2 \tag{28}$$

$$+ \frac{2Mr}{r^2 + \alpha^2 \cos^2\theta}(dt + dr + \alpha \sin^2\theta d\phi)^2 \tag{29}$$

- **Perturbations** of rotating black holes are described by the **Teukolsky equation**, which is a **separable PDE** in the angular and radial part [1], even though the background is neither spherically symmetric nor static. [2]
- Both the angular and radial Teukolsky equation have the same singularity structure as the **confluent Heun equation**. [1]

Kerr BH / $SU(2)$ $N_f = 3$ $\mathcal{N} = 2$ SYM - II: angular part Teukolsky

- **Angular part Teukolsky** equation:

$$\left[\frac{d}{dx}(1-x^2)\frac{d}{dx} + (cx)^2 - 2csx + {}_sA_{lm} + s - \frac{(m+sx)^2}{1-x^2}\right] {}_sS_{lm}(x) = 0 \qquad (30)$$

  $x = \cos\theta$, s is the (minus of) spin, $l = 0, 1, 2, \ldots$, $|m| \leq l$, $m \in \mathbb{Z}$ for integer spins, $m \in \frac{1}{2} + \mathbb{Z}$ for half integer spins, $c = \alpha\omega$.
- ${}_sS_{lm}(x)$ is the **spin-weighted spheroidal harmonics**, with **eigenvalue** ${}_sA_{lm}$ determined by the regularity condition of ${}_sS_{lm}(x)$ at $x = \pm 1$. For general s, l, m, c, no closed form of ${}_sA_{lm}$ is known so far. ${}_sA_{lm}(c = 0) = {}_sA^{(0)}_{lm} = l(l+1) - s(s+1)$.
- By changing $z = (1+x)/2$, $y(z) = \sqrt{1-x^2}\,{}_sS_{lm}(x)/2$ and

$$\Lambda_3 = 16c\,, \quad E = -{}_sA_{lm} - s(s+1) + c^2 - \frac{1}{4} \qquad (31)$$

$$m_1 = -m\,, \quad m_2 = m_3 = -s\,. \qquad (32)$$

  we still connect to the $SU(2)$, $N_f = 3$ **quantum SW curve** equation. [1]

Kerr BH / $SU(2)$ $N_f = 3$ $\mathcal{N} = 2$ SYM - III: quant. cond. ang. Teukolsky

- The **quantization condition** this time is **also** for the A-**period** and reads [1]

$$\Pi_A^{(3)}\left(-{}_sA_{lm} - s(s+1) - c^2 - \frac{1}{4}, \mathbf{m}, 16c, 1\right) = i\left(l + \frac{1}{2}\right) \tag{33}$$

- The **difference between the A-period condition and the B-period condition** is the following. In the angular problem we impose the boundary conditions at the two regular singular points $z = 0, 1$. In the radial problem, we have to impose the conditions at the regular singular point $z = 1$ and at the irregular singular point $z = 1$.
- Equation (33) is **equivalent** to [1]

$$ {}_sA_{lm} - {}_sA_{lm}^{(0)} + c^2 = \Lambda_3 \frac{\partial \mathcal{F}_{\text{inst}}^{(3)}}{\partial \Lambda_3}(il + i/2, \mathbf{m}, \Lambda_3, 1)\Big|_{\Lambda_3 = 16c} \qquad \mathbf{m} = \{-m, -s, -s\} \tag{34}$$

from which we can **obtain** ${}_sA_{lm}$ **as an expansion in powers of** c.

Kerr BH / $SU(2)$ $N_f = 3$ $\mathcal{N} = 2$ SYM - IV: radial part Teukolsky

- **Radial part Teukolsky** equation:

$$\Delta(r)R''(r) + (s+1)\Delta'(r)R'(r) + V_T(r)R(r) = 0 \qquad \Delta(r) = r^2 - 2Mr + \alpha^2 \tag{35}$$

$$V_T(r) = \frac{K(r)^2 - 2is(r-M)K(r)}{\Delta(r)} - {}_sA_{lm} + 4is\omega r + 2\alpha m\omega - \alpha^2\omega^2 \quad K(r) = (r^2+\alpha^2)\omega - \alpha m$$

- Equation (35) has regular singular points at $r = r_\pm = M \pm \sqrt{M^2 - \alpha^2}$, corresponding to the event and Cauchy horizons. Boundary conditions:

$$R(r) \sim \begin{cases} (r_+ - r_-)^{-1-s+i\omega+i\sigma_+} e^{i\omega r_+}(r - r_+)^{-s-i\sigma_+} & r \to r_+ \\ A(\omega) r^{-1-2s+i\omega} e^{i\omega r} & r \to \infty \end{cases} \qquad \sigma_+ = \frac{\omega r_+ - \frac{\alpha m}{2M}}{\sqrt{1 - \frac{\alpha^2}{M^2}}}$$

- **We get** $SU(2)$ $N_f = 3$, by changing $z = (r - r_-)/(r_+ - r_-)$, $y(z) = \Delta(r)^{(s+1)/2}R(r)$, [1]

$$\Lambda_3 = -16i\omega\sqrt{M^2 - \alpha^2} \quad E = -{}_sA_{lm} - s(s+1) + (8M^2 - \alpha^2)\omega^2 - \frac{1}{4} \tag{36}$$

$$m_1 = s - 2iM\omega\,, \quad m_3 = -s - 2iM\omega\,, \quad m_2 = i(-2M^2\omega - \alpha m)/\sqrt{M^2 - \alpha^2}\,. \tag{37}$$

Kerr BH / $SU(2)$ $N_f = 3$ $\mathcal{N} = 2$ SYM - III: quant. cond. rad. Teukolsky

- The radial equation has the **discrete set of complex frequencies** $\omega_n(l, s, m)$ - **QNMs** $M\omega_n$ - given by the **quantization condition for the** B**-period**

$$\Pi_B^{(3)}\left(-{}_sA_{lm} - s(s+1) + (8M^2 - \alpha^2)\omega^2 - \frac{1}{4}, \mathbf{m}, -16i\omega\sqrt{M^2-\alpha^2}, 1\right) = 2\pi\left(n + \frac{1}{2}\right), \tag{38}$$

$$\mathbf{m} = \left\{s - 2iM\omega, \frac{i(-2M^2\omega - \alpha m)}{\sqrt{M^2-\alpha^2}}, -s - 2iM\omega\right\}, \tag{39}$$

where ${}_sA_{lm}$ are given by (34).

- Thus through the combination of the angular and radial quantization condition it is possible to **find both** ${}_sA_{lm}$ **and** ω_n **by using quantities in** $\mathcal{N} = 2$ **SW theory**. [1]

Extremal limit Kerr BH / $SU(2)$ $N_f = 2$ $\mathcal{N} = 2$ SYM - I

- The **extremal limit in Kerr BHs** corresponds to $\alpha \to M$, which means in gauge theory

$$\Lambda_3 \to 0\,, \quad m_2 \to \infty\,, \quad \Lambda_3 m_2 = -16M\omega(2M\omega - m) = \text{fixed}\,, \tag{40}$$

  that is, the **decoupling limit under which the $N_f = 3$ theory flows to the $N_f = 2$ one**. [1]

- The radial part of the Teukolsky equation corresponds then to the **double confluent Heun** equation, quantum SW curve of $SU(2)$ $N_f = 2$ **SYM** under the dictionary: [1]

$$\Lambda_2^2 = -16M\omega(2M\omega - m) \qquad E = -{}_sA_{lm} - s(s+1) + 7M^2\omega^2 - \frac{1}{4} \tag{41}$$

$$m_1 = s - 2iM\omega \qquad m_2 = -s - 2iM\omega\,. \tag{42}$$

Extremal limit Kerr BH / $SU(2)$ $N_f = 2$ $\mathcal{N} = 2$ SYM - II

- In the extremal limit, the **radial quantization condition** for the B-period for the $N_f = 2$ theory **gets simplified** [1]

$$\Pi_B^{(2)}\left(-{}_sA_{lm} - s(s+1) + 7M^2\omega^2 - \frac{1}{4}, \mathbf{m}, -4i\sqrt{\omega(2M^2\omega - Mm)}, 1\right) = 2\pi\left(n + \frac{1}{2}\right), \tag{43}$$

$$\mathbf{m} = \{s - 2iM\omega, -s - 2iM\omega\} \tag{44}$$

(A)dS Schwarschild BH / $SU(2)$ $N_f = 4$ & $SU(2) \times SU(2)$ quiver $\mathcal{N} = 2$ SYM

- Consider **4D asymptotically (A)dS Schwarschild BHs**. The radial master equation has the same form as (13) with

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2, \quad V(r) = f(r)\left[\frac{l(l+1)}{r^2} + (1-s^2)\left(\frac{2M}{r^3} - \frac{4-s^2}{6}\Lambda\right)\right] \tag{45}$$

where Λ is the cosmological constant. So after the redefinition of $\phi(r) = \Phi(r)/\sqrt{f(r)}$ [1]

$$\Phi''(r) + \left[\frac{\omega^2 - V(r)}{f(r)^2} + \frac{f'(r)^2}{4f(r)^2} - \frac{f''(r)}{2f(r)}\right]\Phi(r) = 0 \tag{46}$$

- The 3 roots of $f(r) = 0$ as well as $r = 0$ are regular singular points. For $s = 1, 2$, $r = \infty$ is not a singular point, but for $s = 0$ it is a regular singular point.
- **For** $s = 1, 2$ the master eq. is a **Heun equation** and it corresponds to **gauge theory with** $N_f = 4$. The singularity structure **for** $s = 0$ is realized in an $SU(2) \times SU(2)$ **quiver gauge theory**. [1]

Higher dimensional (A)dS / $SU(2)^{D-2}$ & $SU(2)^{D-3}$ quiver gauge theories

- [1] consider also **higher D-dimensional asymptotically (A)dS BHs**.
  - **For $D > 4$ and $\Lambda \neq 0$** the quantum SW curve having the same singularity structure as the master equation (**Fuchsian ODE with $D+1$ singular points**) is an $SU(2)^{D-2}$ **quiver gauge theory**.
  - **For $\Lambda = 0$** there is an **irregular singular point** with Poincaré rank one, so we have to consider $SU(2)^{D-3}$ **quiver gauge theories** associated with the **Riemann sphere with an irregular puncture**.
- We notice that the **dimensional information of the black hole** is reflected in the **number of quiver gauge groups**. [1]

D3 brane BH / $SU(2)$ $N_f = 0$ $\mathcal{N} = 2$ SYM - I

- **D3 brane** metric

$$ds^2 = H(r)^{-1/2}(-dt^2 + d\mathbf{x}^2) + H(r)^{1/2}(dr^2 + r^2 d\Omega_5^2) \tag{47}$$

where $\mathbf{x}$ are the longitudinal coordinates, $H(r) = 1 + \frac{L^4}{r^4}$, $d\Omega_5^2$ denotes the metric of the transverse round S_5 sphere.

- The master equation in the variable $z = -i \ln \frac{r}{L}$ is the **Mathieu equation** [3]

$$\Phi''(z) + [\alpha - 2\beta \cos(2z)]\, \Phi(z) = 0\,. \tag{48}$$

- The parameter change to the $SU(2)$ $N_f = 0$ **gauge theory** is [3]

$$\alpha = \frac{4u}{\hbar^2} = (l+2)^2\,, \quad \beta = \frac{4\Lambda^2}{\hbar^2} = \omega^2 L^2\,. \tag{49}$$

D3 brane BH / $SU(2)$ $N_f = 0$ $\mathcal{N} = 2$ SYM - II

- The *A*-**period** is given by the **Floquet exponent**

$$a(\hbar, u, \Lambda) = \frac{\hbar}{2}\nu(\alpha, \beta)\,. \tag{50}$$

- The QNMs $\omega(l, n)$ can be obtained by solving numerically the **quantization condition** [3]

$$\nu = \frac{n}{2}\,. \tag{51}$$

- **Particular limits** of the Floquet exponent can be obtained from the expansions for α

$$\alpha = \nu^2 + \frac{\beta^2}{2(\nu^2-1)} + \frac{(5\nu^2+7)\beta^4}{32(\nu^2-1)^3(\nu^2-4)} + \ldots \qquad \beta \ll \alpha \quad \text{(weak coup.)} \tag{52}$$

$$\alpha = -2\beta + 4\nu\sqrt{\beta} - \frac{1}{8}(1+4\nu^2) - \frac{\nu(4\nu^2+3)}{2^6\sqrt{\beta}} + \ldots \quad \alpha \simeq \pm 2\beta \quad (u \simeq \pm 2\Lambda^2) \quad \text{(photon sp.)} \tag{53}$$

General picture

Beginnings of a general picture I

- [3] and [6] extend what they call the SW-QNM correspondence to a **large class of gravity backgrounds** and develop methods that allow to test the QNM results.
- [3] consider **classical integrable geometries** that allow to write down separate, but some times inter-twined, ODEs, **generalising the Teukolsky equations**.
  - [3] find that radial and angular equations can be mapped to $\mathcal{N} = 2$ **theories with a single** $SU(2)$ **gauge group factor**, possibly with hyper-multiplets in the fundamental.
  - The **dictionary is not one - to one**: a single gravity background can be described in terms of apparently unrelated gauge theories with different number of flavours related by modular transformations of the underlying elliptic geometry [3].
  - [3, 6] consider:
    - Kerr-Newman BHs in D $=$ 4 asymptotically flat or AdS;
    - CCLP geometries describing charged and rotating solutions of Einstein-Maxwell theory in D $=$ 5;
    - Myers-Perry and BMPV BHs;
    - D3-branes and their bound-states,

Beginnings of a general picture I (cont.)

    - D1D5 circular fuzzballs.

- [3] consider for simplicity only **neutral scalar perturbations**. Metric and vector perturbations lead to similar equations.
- The **common feature** of the examples considered by [3] is the existence of a **photon-sphere**, that traps light on unstable null orbits eventually leaking out in ringdown modes. They find that photon-spheres are associated to degeneration of the quantum elliptic SW geometry where a **cycle pinches to zero size in the semiclassical eikonal limit** $\hbar \to 0$. The singularity is smoothed out by quantum correction once $\epsilon_1 = \hbar$ is turned on [6].

Beginnings of a general picture II - General ODE

- [3, 6] find that **QNMs** are encoded in **canonical 2nd order ODEs** with coefficient a **ratio of polynomials of degree** $(2n+2)$ **and** $(n+3)$:

$$\Psi'(z) + Q(z)\Psi(z) = 0 \qquad Q(z) = \frac{P_{2n+2}(z)}{\Delta_{n+3}(z)^2}\,. \tag{54}$$

  The same differential equation describes the dynamics of an $SU(2)^n$ linear $\mathcal{N}=2$ quiver theory in the NS limit $\epsilon_1 = \hbar, \epsilon_2 = 0$.

- More precisely, the Q-function defines the quadratic differential $\phi_2 = Q(z)dz^2$ of the Ω-deformed version of the SW. Zeroes of $P_{2n+2}(z)$ specify the positions of the branch points of the associated SW curve, while those of $\Delta_{n+3}(z)$ encode the gauge couplings. Identifying the two Q's one can establish a dictionary between the parameters describing the gravity solution (radial/angular variable z, mass M, charge Q, angular momentum J , frequency ω and conserved 'quantum' numbers l, m's) and the gauge theory parameters ($z_S W$, the RG scale Λ, hypermultiplet masses m_f and Coulomb branch moduli u_a). [6]

Beginnings of a general picture III - Asymptotic analytic QNMs

- Consider a **D3 brane**. The **photon sphere** at $r = r_c$ is where lights gets trapped orbiting forever for a specific choice of frequency ω_c. By considerations on the Hamiltonian for the motion of massless neutral probes [3] find the **asymptotics**

$$r_c = L \,, \quad \omega_c \simeq \frac{l+2}{\sqrt{2}L} \,, \quad \text{for} \quad r \simeq r_c \,, \quad l \to \infty \,. \tag{55}$$

- Consider a **massless scalar field in a D3 brane** background.
  - After some change of wave function its radial wave equation is of the form (54) with [3]

$$Q(r) = \frac{4\omega^2(r^4 + L^4) - r^2[4l(l+4)+15]}{4r^4} \,. \tag{56}$$

Beginnings of a general picture III - Asymptotic analytic QNMs (cont.)

- In the **semiclassical limit**, where ω, l are large, the QNMs follow from the Bohr-Sommerfeld quantization condition (with $r_\pm$ the inversion points, where $Q(r_\pm) = 0$) [3]

$$\int_{r_-}^{r_+} \sqrt{Q(r)}\, dr = \pi \left(n + \frac{1}{2} \right) . \tag{57}$$

- The integral (57) can be approximated by expanding $Q(r)$ around its minimum at r_c to quadratic order leading to

$$\frac{Q(r_c)}{\sqrt{2\partial_r^2 Q(r_c)}} = -i \left(n + \frac{1}{2} \right) . \tag{58}$$

This equation can be solved by giving a **small imaginary part to** ω, i.e. writing $\omega = \omega_R + i\omega_I$ and solving perturbatively in ω_I. To linear order in ω_I one finds

$$\omega \simeq \frac{1}{\sqrt{2}L} \sqrt{(l+2)^2 - \frac{1}{4}} - \frac{i}{2L}(2n+1) , \tag{59}$$

in agreement with the geodetic results (1), (55) at large l. [3]

Beginnings of a general picture III - QNMs from quantum SW curves I

- The **dynamics of $\mathcal{N}=2$ gauge theory** with fundamental matter in a non-trivial **NS-background** can be described by the **differential equation** [3]

$$\left[\Lambda^{4-N_f} y^{\frac{1}{2}} P_+(x) y^{\frac{1}{2}} + P_0(x) + y^{-\frac{1}{2}} P_-(x) y^{-\frac{1}{2}}\right] \Psi = 0\,. \tag{60}$$

$$P_+(x) = \prod_{i=1}^{N_+}(x-m_i)\,,\; P_-(x) = \prod_{i=N_++1}^{N_f}(x-m_i)\,,\; P_0(x) = x^2 - u + p(x)\,,\; u = \frac{1}{2}\langle \mathrm{tr}\,\varphi^2\rangle \tag{61}$$

with $p(x) = \Lambda^{4-N_f}(x - \sum_i m_i + \frac{\hbar}{2})$ for $N_f = 3$, $p(x) = \Lambda^{4-N_f}$ for $N_f = 2$ and $p(x) = 0$ otherwise.

Beginnings of a general picture III - QNMs from quantum SW curves II

- In equation (60) Ψ is an energy eigenstate and x, y are operators such that

$$[x, \ln y] = \hbar\,. \tag{62}$$

- The **gauge/gravity dictionary** follows from [3]

$$Q(y) = Q(z)z'(y)^2 + \frac{z'''(y)}{2z'(y)} - \frac{3}{4}\left[\frac{z''(y)}{z(y)}\right]^2\,. \tag{63}$$

Beginnings of a general picture III - QNMs from quantum SW curves III

- One can view (60) as an **ODE of 2nd order** in y by setting $\hat{x} = \hbar y \partial_y$ or as a **difference equation** in x by setting $y = e^{-\hbar \partial_x}$. [3]
- For instance, using $P(x)y = yP(x+\hbar)$

$$0 = \left[A(y)\hat{x}^2 + B(y)\hat{x} + C(y)\right]\Psi(y) \tag{64}$$

$$= \left[\Lambda^{4-N_f} y^2 P_+(\hat{x} + \frac{\hbar}{2}) + yP_0(\hat{x}) + P_-(\hat{x} - \frac{\hbar}{2})\right]\Psi(y) \tag{65}$$

bringing this equation to canonical form [3] find

$$Q_{SW}(y) = \frac{4CA - B^2 + 2\hbar y(BA' - AB') + \hbar^2 A^2}{4\hbar^2 y^2 A^2}. \tag{66}$$

Beginnings of a general picture III - QNMs from quantum SW curves IV

- Alternatively, viewing (60) as a difference equation one can write the $\hbar$-deformed SW equation

$$\Lambda^{4-N_f} M(x) W(x) W(x-\hbar) + P_0(x) W(x) + 1 \tag{67}$$

$$M(x) = P_+(x - \frac{\hbar}{2}) P_-(x - \frac{\hbar}{2}) \qquad W(x) = \frac{1}{P_-(x+\frac{\hbar}{2})} \frac{\hat{\Psi}(x)}{\hat{\Psi}(x+\hbar)} \tag{68}$$

which can be recursively solved order by order in $q = \Lambda^{4-N_f}$. [3]

2D CFT / Kerr BHs correspondence

AGT duality for ODE

- According to **AGT duality**, **conformal blocks** of Virasoro algebra for **2D CFT** can be identified with concrete combinatorial formulae arising from equivariant **instanton counting** in the context of $\mathcal{N} = 2$ **4D supersymmetric gauge theories**.
- 2D CFT and of the representations of its Virasoro symmetry algebra provide **exact solutions to PDEs** in terms of conformal blocks and appropriate fusion coefficients. [4]
    - The prototypical example is the **null-state equation at level** 2 for primary operators of Virasoro algebra which reduce, in the large central charge limit, to a **Schrödinger-like equation** with regular singularities, corresponding to a potential term with at most quadratic poles. [4]
    - The wave function $\Psi(z)$ corresponds to the insertion of a BPS surface observable in the gauge theory path integral. The relevant gauge theory moduli space in these cases is the one of ramified instantons, with vortices localised on the surface defect, the z-variable providing the fugacity for the vortex counting.
    - The advantage of this approach is that the **explicit solution of the connection problem** on the z-plane for the equation can be derived from the explicit computation of the **full** CFT_2 **correlator** and from its **expansions in different intermediate channels**. [4]

Confluent Heun equation as BPZ equation

- The representation theory of the Virasoro algebra for central charge $c = 1 + 6Q^2$, $Q = b + \frac{1}{b}$, contains degenerate Verma modules of weight $\Delta_{r,s} = \frac{Q^2}{4} - \alpha_{r,s}^2$ with $\alpha_{r,s} = -\frac{br}{2} - \frac{s}{2b}$.
- At level 2, the degenerate field $\Phi_{2,1}$ satisfies the **null state equation**

$$(b^{-2}L_{-1}^2 + L_{-2})\Phi_{2,1}(z) = 0 \qquad (69)$$

  It translates into a differential equation, the **BPZ equation for correlation functions**.
- Consider the **chiral correlator with a degenerate field insertion**

$$\Psi(z) = \langle \Delta, \Lambda_0, m_0 | \Phi_{2,1}(z) V_2(1) | \Delta_1 \rangle \qquad (70)$$

  where $|\Delta_1\rangle$ is a primary state and $\langle \Delta, \Lambda_0, m_0|$ is an irregular state of rank 1 [4].

Confluent Heun equation as BPZ equation (cont.)

- On the resulting BPZ equation, we take the **Nekrasov-Shatashvili (NS) limit** in the AGT dual gauge theory, corresponding to the **semiclassical limit of large Virasoro central charge $c \to \infty$ in the CFT**. To this end, we introduce a new parameter $\hbar$, send $\epsilon_2 = \hbar b \to 0$ and keeping $\epsilon_1 = \hbar / b$, $\hat{\Delta} = \hbar^2 \Delta$, $\hat{\Delta}_k = \hbar^2 \Delta_k$, $\Lambda = 2i\hbar\Lambda_0$, $m_3 = \frac{i}{2}\hbar m_0$ fixed. Introducing the normalized wave function $\psi(z) = \lim_{\epsilon_2 \to 0} \Psi(z) / \langle \Delta, \Lambda_0, m_0 | V_2(1) | \Delta_1 \rangle$ the BPZ equation becomes

$$\left[\epsilon_1^2 \partial_z^2 - \frac{1}{z}\frac{1}{z-1}(-\Lambda\partial_\Lambda \mathcal{F}^{inst} + \hat{\Delta}_2 + \hat{\Delta}_1 - \hat{\Delta}) + \frac{\hat{\Delta}_2}{(z-1)^2} + \frac{\hat{\Delta}_1}{z^2} - \frac{m_3\Lambda}{z} - \frac{\Lambda^2}{4}\right]\psi(z) = 0 \tag{71}$$

Confluent Heun equation as BPZ equation (cont.)

- It takes the form of a **Schrödinger equation**

$$\epsilon_1^2 \frac{d^2\psi(z)}{dz^2} + \frac{1}{z^2(z-1)^2} \sum_{i=0}^{4} A_i z^i = 0 \tag{72}$$

Using conformal momenta instead of dimensions we write $\hat{\Delta}_i = \frac{1}{4} - a_i^2$. with $a_i = \hbar\alpha_i$. Defining furthermore $E = a^2 - \Lambda\partial_\Lambda \mathcal{F}^{inst}$ the coefficients of the potential are

$$A_0 = \frac{\epsilon_1^2}{4} - a_1^2 \qquad A_1 = -\frac{\epsilon_1^2}{4} + E + a_1^2 - a_2^2 - m_3\Lambda \tag{73}$$

$$A_2 = \frac{\epsilon_1^2}{4} - E + 2m_3\Lambda - \frac{\Lambda^2}{4} \qquad A_3 = -m_3\Lambda + \frac{\Lambda^2}{2} \qquad A_4 = -\frac{\Lambda^2}{4} \tag{74}$$

Confluent Heun equation as BPZ equation (cont.)

- The **dictionary** for connecting to the **radial Teukolsky** for **Kerr BHs** equation is

$$E = \tag{75}$$

$$a_1 = -i\frac{\omega - m\Omega}{4\pi T_H} + 2iM\omega + \frac{s}{2} \qquad a_2 = -i\frac{\omega - m\Omega}{4\pi T_H} - \frac{s}{2} \tag{76}$$

$$m_3 = -2iM\omega + s \qquad \Lambda = -2i\omega(r_+ - r_-) \tag{77}$$

where T_H is the Hawking temperature and Ω is the angular velocity at the horizon given by

$$T_H = \frac{r_+ - r_-}{8\pi M r_+}\,, \qquad \Omega = \frac{a}{2Mr_+}\,. \tag{78}$$

Confluent Heun equation as BPZ equation (cont.)

The remaining masses are obtained by AGT

$$m_1 = a_1 + a_2 = -i\frac{\omega - m\Omega}{2\pi T_H} + 2iM\omega \tag{79}$$

$$m_2 = a_2 - a_1 = -2iM\omega - s \tag{80}$$

$$m_3 = -2iM\omega + s\,. \tag{81}$$

- On the other hand, the dictionary which connects to **angular Teukolsky** equation for **Kerr BHs** is

$$E = \frac{1}{4} + c^2 + s(s+1) - 2cs + \lambda \tag{82}$$

$$a_1 = -\frac{m-s}{2} \qquad a_2 = -\frac{m+s}{2} \tag{83}$$

$$m_3 = -s \qquad \Lambda = 4c \tag{84}$$

Confluent Heun equation as BPZ equation (cont.)

and all the masses of gauge theory are, by AGT

$$m_1 = a_1 + a_2 = -m\,, \quad m_2 = a_2 - a_1 = -s\,, \quad m_3 = -s\,. \tag{85}$$

The connection problem

- Exploiting **crossing symmetry of Liouville correlation functions** we can **connect different asymptotic expansions** of the solutions of BPZ equations around different field insertion points. [4]
- The irregular correlators appearing in such asymptotics can be efficiently computed as **Nekrasov partition functions** thanks to the AGT correspondence (in the NS limit $\epsilon_2 \to 0, \epsilon_1 = 1$).
- In particular, the connection matrix M is defined as the matrix which relates solution of the confluent Heun equation around 1 and ∞ (t and u channel) which give the irregular correlator

$$f_i^{(t)}(z) = M_{ij} f_j^{(u)}(z) \qquad i = \alpha_{2\pm}\,, \quad j = \alpha_{\pm}\,. \tag{86}$$

- There are several interesting **physical quantities in the black hole** problem which are governed by the Teukolsky equation. Having the **explicit expression for the connection coefficients** allows [4] to **compute them exactly**. See next subsection and section 6.

Proof of quantization conditions

- From the explicit expression of the connection matrix M, [4] extract the quantization condition for the quasinormal modes, by **imposing that the coefficient of the ingoing wave vanishes**.

$$\partial_a \mathcal{F} = 2\pi i n\,, \qquad n \in \mathbb{Z}\,, \tag{87}$$

  which corresponds to (38).

- [4] compute also of the angular eigenvalue. To that end, they impose that the **angular eigenfunctions be regular at** $z=0,1$, relative to another connection matrix N. The divergences are cured if

$$a = -i\Pi_A^{(3)} = l + \frac{1}{2}\,, \qquad l \geq \max\left(m, s\right), \tag{88}$$

  which corresponds to (33).

New results for Kerr(-dS) BHs

New result from $\mathcal{N} = 2$ SYM: an alternative to Kerr Teukolsky equation

- Consider **Kerr BHs**. The **radial Teukolsky equation** is complicated and uncoventional. This fact often prevents us from understanding its analytic properties. [2]
- [2] found a **much simpler** description through a **new alternative equation**

$$\left[f(z)\frac{d}{dz}f(z)\frac{d}{dz} + (2M\omega)^2 - V(z)\right]\phi(z) = 0 \quad \text{with} \quad f(z) = 1 - \frac{1}{z} \tag{89}$$

$$V(z) = f(z)\left[4c^2 + \frac{4c(m-c)}{z} + \frac{{}_sA_{lm} + s(s+1) - c(2m-c)}{z^2} - \frac{s^2-1}{z^3}\right] \tag{90}$$

- The alternative equation (89) is **isospectral** to the original one. Besides, it has the same form as the master equations for spherically symmetric BHs if c and ${}_sA_{lm}$ are regarded as given parameters [2].
- Once moving to the $\mathcal{N} = 2$ SYM, we have a **symmetry for flavor masses** respected by the **QNM spectrum**. However, the same symmetry is **not manifest** at the level of differential equations. [2]
- The **physical origin** of equation (89) in BH perturbation theory is **not known**. [2]

General modal stability problem for Kerr(-dS) BHs

- As **Kerr(-de Sitter)** black holes are stationary spacetimes, they correspond to equilibrium states for Einsten equations, and one would like to **determine** whether they are **stable or unstable equilibria**. [5]
- Outside very special classes, stability of Kerr(-de Sitter) is an **open question**. [5]
- If Kerr-(dS) BHs are nonlinearly stable, the most basic statement one can hope to prove for Teukolsky equation is that it is **modally stable**, i.e. that there are no separable solutions to it which are exponentially growing or bounded and non-decaying in time. By separable solution we mean a solution of the form

$$\alpha^{[s]}(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} S(\theta) R(r) , \tag{91}$$

  where S and R satisfy, respectively, the angular and radial Teukolsky ODE. [5]
- For Kerr-(dS) BHs there are **many possible sources of mode instabilities** (such as *superradiance*). [5]
  - It is remarkable that, in the $\Lambda = 0$ **case, mode stability holds for the entire Kerr black hole family** with angular momentum $|a| \leq M$.
  - In the $\Lambda > 0$ **Kerr-dS** case modal stability remains an **open problem** for general parameters.

New result from $\mathcal{N} = 2$ SYM: proofs of mode stability for Kerr(-dS) BHs

- **Thanks to the correspondence with $\mathcal{N} = 2$ theories** discovered first by [1], [5] gave
  1. a new alternative proof of the classic mode stability result for Kerr BHs
  2. a **completely new (though partial in the parameter range) proof of mode stability for Kerr-dS BHs**.
- The proofs of [5] makes use of **previously unknown symmetries of the point spectrum** of the radial Teukolsky equation conjectured by [1, 2] and proven by [5].
- The second proof of [5] is most useful in the case $s \in \mathbb{Z}$, where it rules out some of the modes in the superradiant range, where it is known from energy conservation that unstable and non-decaying modes, if they exist, must lie. [5]

New result from 2D CFT: greybody factor

- The emission spectrum as measured by an observer at infinity is no longer thermal, but is given by

$$\frac{\sigma(\omega)}{\exp\frac{\omega - m\Omega}{T_H} - 1} \tag{92}$$

where $\sigma(\omega)$ is the so-called **greybody factor**, which can be defined as the flux going into the horizon normalized by the flux coming in from infinity $\sigma = \phi_{abs}/\phi_{in}$ [4].

- From the **solution of the connection problem** for the radial Teukolsky / confluent Heun equation, [4] obtain an exact expression of the **greybody factor** σ in terms of the instanton part of the **NS free energy** $\mathcal{F}^{inst}$.
  - Their result can be expressed in terms of only **BH parameters** by using **Matone's relation** (9).
  - They **checked their result with two different asymptotics**. One of them is the semiclassical result obtained by **WKB analysis of the ODE**, where it is given by the **dual SW period**

$$\sigma \simeq \exp\left\{-\frac{a_D}{\epsilon_1}\right\}, \qquad \epsilon_1 \to 0\,. \tag{93}$$

New result from 2D CFT: Love numbers

- Applying an **external gravitational field to a self-gravitating body** generically causes it to deform. The response of the body to the external gravitational tidal field is captured by the so-called **tidal response coefficients** or **Love numbers** [4].
- In general relativity, the tidal response coefficients are generally complex, and the **real part captures the conservative response** of the body, whereas the **imaginary part captures dissipative effects**. For four-dimensional Kerr black holes, the conservative (real part) of the response coefficient to static external perturbations has been found to vanish.
- Love numbers are **measurable quantities** that can be probed with **gravitational wave observations**. [4]
- **Using their 2D CFT approach** to the Teukolsky equation [4] compute
  - the **static** $\omega = 0$ **Love number** which agrees with the result in the literature;
  - the **non-static Love number** as an expansion in ω, that improves the result in the literature by also adding instanton corrections. [4]

Possible future developments

Possible future developments

- In would be interesting to investigate how the existence of **unstable BH solutions is reflected** on the **SW theory** side. [1]
- [1] focused on uncharged BHs, but they expect that their analysis to carry on for **Reissner-Nordström and Kerr-Newman BHs** as well.
- Even though the Nekrasov-Shatashvili free energy is a convergent series in the instanton counting parameter, the convergence is a bit slow. From that perspective it would be good to **compute these quantum periods** by using the alternative **TBA approach** [1].
- It would be interesting to **study the singularities in the Borel plane** (used for resummation of WKB gauge periods) and see if there is any **interpretation as wall crossing phenomena** from the black hole viewpoint. [1]

Possible future developments (cont.)

- [1] result also indicates that it should be possible to **compute efficiently the WKB expansion for the QNMs** by using the **holomorphic anomaly equation** for the underlying gauge theory. It would be interesting to investigate this aspect more in detail as a possible alternative to the existing approaches. [1]
- There is **another third order differential equation** in the three-flavored SQCD. Probably this equation is also isospectral to the Teukolsky equation. In this third order equation, the mass symmetry is manifest. It is interesting to study it in detail. [2]
- The result of [2] could be **extended to higher dimensional or asymptotically non-flat cases**.
- Near the **extremal limit** the QNM spectrum shows a transition between two branches called damped modes and zero-damping modes. However, the second mass in (37) does not diverge in $\alpha \to M$ for the zero-damping modes. The author of [2] claims to not **understand its physical interpretation**.

Possible future developments (cont.)

- The potential (90) with $s = 2$ simply reduces to the Regge-Wheeler potential in the limit $c \to 0$. The Regge-Wheeler potential is obtained by the odd-parity gravitational perturbation, while the even-parity perturbation leads to the Zerilli potential. It is unclear why the dual equation of [2] is a deformation of the Regge-Wheeler potential rather than of the Zerilli potential. Is there a **further equivalent description based on the Zerilli potential?**
- From the 2D CFT perspective, the equation (47) arises in the semiclassical limit of Liouville field theory. An intriguing question to investigate is **whether the quantum corrections in 2D CFT** can have a **physical interpretation in the black hole description**. In principle, this could be related to quantum gravitational corrections or more generally to some **deviations from General Relativity**. [4]

Possible future developments (cont.)

- Another link of holographic type between 2D CFT and Kerr black hole physics emerged in the last years: the **Kerr/CFT correspondence**. It would be very interesting to find whether the mathematical structure behind the solution of the Kerr black hole radiation problem presented by [4] could have a clear **interpretation** in the context of the Kerr/CFT correspondence.
- A further possible application of the method presented by [4] is the study of the **physics of the last stages of coalescence of compact objects** with the Zerilli function.
- **Other black hole backgrounds** can be analysed with methods similar to the ones used in [4]. An important example is given by Kerr black hole solutions which asymptote to the (Anti-)de Sitter metric at infinity. These correspond to the Heun equation, which has four regular singularities on the Riemann sphere, and can be engineered from five-point correlators in Liouville CFT with four primary operator insertions and one level 2 degenerate field.

Possible future developments (cont.)

- The method of [4] can well be extended to **other gravitational potentials** studied to **analyse possible deviations from GR** with a modified quasinormal mode spectrum and Love numbers.
- The results [4] present are given as a perturbative series in the instanton counting parameter Λ, which actually converges very efficiently. Understanding how to **extend their approach to the connection problem on the Λ plane** would improve the understanding the **strong coupling effects in gauge theory**. Moreover, it could reveal to be useful for other applications in gravitational problems. [4]
- **M-theory** may provide a **hint on the physical and geometrical origin** of this correspondence among 4D gauge theory and BH (or brane) solution [6].

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