Integrability as a new method for exact results on quasinormal modes of black holes

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Outline

Outline (cont.)

4. Comparison of methods of computation of quasinormal modes

5. Generalization and perspectives

Introduction

On quasinormal modes of black holes

- Quasinormal modes (QNMs) are in some sense the "characteristic sound" of black holes, that is harmonic oscillations representing the time evolution of some initial displacement from their equilibrium state.
- During a certain time interval, the evolution of some initial perturbation is dominated by damped single-frequency oscillations.
- The frequency and damping of these oscillations depend only on the parameters of the black hole.
- They essentially involve the spacetime metric outside the horizon, but also possible matter fields.

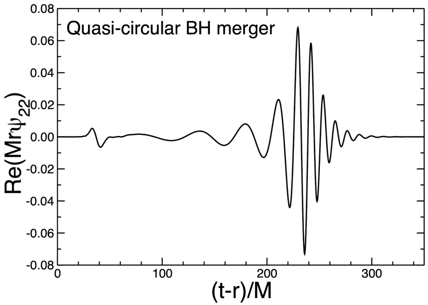
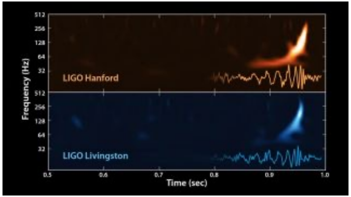
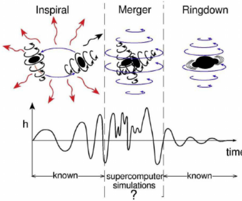


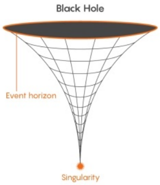
Figure: (Berti-Cardoso-Starinets:2009)

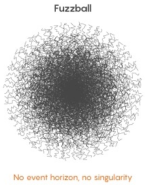
Quasinormal modes and gravitational waves

- Quasinormal modes are seen to dominate the gravitational radiation wave forms at certain times (ringdown phase), even though they are a concept born from linearized perturbation calculations.
- BH QNMs can be used to infer their mass and angular momentum and to test the no-hair theorem of general relativity.
- At later ringdown stages, Exotic Compact Objects (ECOs) of modified theories of gravity and fuzzballs of String Theory produce significant deviations from GR. The crucial role played by QNMs in discriminating BHs from fuzzballs or other ECOs motivated renewed effort in their determination with higher and higher accuracy.









Quasinormal modes and holography

- According to AdS/CFT, equilibrium and non-equilibrium properties of strongly coupled thermal gauge theories are related to the physics of higher-dimensional BHs and black branes and their fluctuations.
- Quasinormal spectra of the dual gravitational backgrounds give the location of the poles of the retarded correlators in the gauge theory, supplying important information about the theory's quasiparticle spectra and transport coefficients.
- The duality offers a new perspective on notoriously difficult problems, such as the BH information loss paradox, the nature of BH singularities and quantum gravity. Holographic approaches to these problems often involve QNMs.

Mathematical definition of quasinormal modes I

- Perturbations of the BH metric or fields can be shown to constitute solutions Φ of some PDE of the form

$$\left\{ +\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + U(x) \right\} \Phi(t,x) = 0\,, \tag{1}$$

  where x here is a coordinate ("tortoise" coordinate) such that the BH horizon is put at $x \to -\infty$ and spacetime infinity at $x \to +\infty$.
- If we take the Laplace transform of Φ

$$\hat{\Psi}(s,x) = \int_0^\infty e^{-st} \Phi(t,x)\, dt\,, \tag{2}$$

$$\left\{ -\frac{\partial^2}{\partial x^2} + U(x) + s^2 \right\} \hat{\Psi}(s,x) = -\mathcal{I}(s,x)\,, \tag{3}$$

  with $\mathcal{I}(s,x) = -s\Psi(t,x)\big|_{t=0} - \frac{\partial \Psi(t,x)}{\partial t}\big|_{t=0}$.

Mathematical definition of quasinormal modes II

- The corresponding homogeneous equation is exactly the ODE we are going to study in the next sections
$$\left\{-\frac{\partial^2}{\partial x^2} + U(x) + s^2\right\}\Psi(s,x) = 0\,. \tag{4}$$
Its solutions bounded at $x \to \pm\infty$, for $\mathrm{Re}\, s > 0$, are
$$\begin{aligned} \Psi_+(s,x) &\sim e^{-sx}\,, \qquad x \to +\infty \\ \Psi_-(s,x) &\sim e^{sx}\,, \qquad x \to -\infty\,. \end{aligned} \tag{5}$$
- The solution of the homogenous equation is then found to be given by the Green function G as
$$\hat{\Psi}(s,x) = \int_{-\infty}^{\infty} G(s,x,x')\mathcal{I}(s,x')dx'\,, \qquad G(s,x,x') = \frac{1}{W[\Psi_-,\Psi_+]}\Psi_-(s,x_<)\Psi_+(s,x_>)\,, \tag{6}$$
with $x_< = \min(x',x)$, $x_< = \max(x',x)$.

Mathematical definition of quasinormal modes III

- Then taking the antiplace transform of $\hat{\Psi}$

$$\Phi(t,x) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{st}\hat{\Psi}(s,x)\,ds\,, \tag{7}$$

we get the original perturbation as

$$\begin{aligned}\Phi(t,x) &= \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{st}\int_{-\infty}^{\infty} G(s,x,x')\mathcal{I}(s,x')\,dx'\,ds \\ &= \frac{1}{2\pi i}\oint e^{st}\frac{1}{W(s)}\int_{-\infty}^{\infty}\Psi_-(s,x_<)\Psi_+(s,x_>)\mathcal{I}(s,x')\,dx'\,ds \\ &= \sum_q e^{s_q t}\mathrm{Res}\left(\frac{1}{W(s)}\right)\Bigg|_{s_q}\int_{-\infty}^{\infty}\Psi_-(s_q,x_<)\Psi_+(s_q,x_>)\mathcal{I}(s_q,x')\,dx'\,.\end{aligned} \tag{8}$$

We notice that the perturbation is written as a sum over the residues of the zeros of the wronskian of the fundamental regular solutions at $x\to\pm\infty$ (5):

Mathematical definition of quasinormal modes IV

- The crucial point for us is that the perturbation is a sum over the residues of the inverse wronskian of the regular solutions (5):

$$W(s_n) = W[\Psi_+, \Psi_-] = 0\,. \tag{10}$$

  Besides, condition (10) means that at these special points the two solutions (5) (in general independent) become linearly dependent.

- By setting $s = i\omega$ we recover the usual intuitive definition of QNMs as the frequencies of plane wave solutions both incoming at the horizon and outgoing at infinity. However, as well explained in [Nollert:1999], this last definition is not mathematically rigorous, since it would lead to diverging boundary conditions. Instead, the QNMs ω_n have an imaginary part $\mathrm{Im}\,\omega_n < 0$, so that the perturbation they describe is damped to zero as $t \to \infty$.

A quantum integrable model for black holes

ODE for the perturbation of generalized RN BHs

- Line element for intersection of four stacks of D3-branes in type IIB supergravity:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]\,, \tag{11}$$

  with

$$f(r) = \prod_{i=1}^{4}\left(1 + \frac{\mathcal{Q}_i}{r}\right)^{-\frac{1}{2}}\,. \tag{12}$$

- If the charges $\mathcal{Q}_i = Q = M$ are all equal, it leads to an extremal Reissner Nordström BH with $f(r) = 1 - \frac{2M}{r} + \frac{\mathcal{Q}^2}{r^2}$.
- The equation describing the scalar perturbation is

$$\frac{d^2\phi}{dr^2} + \left\{-\frac{(l+\frac{1}{2})^2 - \frac{1}{4}}{r^2} + \frac{\omega^2}{r^4}\left[\Sigma_4 + \Sigma_3 r + \Sigma_2 r^2 + \Sigma_1 r^3 + r^4\right]\right\}\phi = 0 \tag{13}$$

  where we defined $\Sigma_k = \sum_{i_1<\ldots<i_k}^{4} \mathcal{Q}_{i_1}\cdots\mathcal{Q}_{i_k}$.
- It is a Doubly Confluent Heun equation with two irregular singularities at $r \to 0, \infty$.

ODE for generalized Perturbed Hairpin Integrable Model

- Changing variable and parameters as

$$r = \sqrt[4]{\Sigma_4} e^y \,, \qquad \omega \sqrt[4]{\Sigma_4} = -ie^\theta \,, \tag{14}$$

$$\frac{\Sigma_1}{\sqrt[4]{\Sigma_4}} = 2M_1 e^{-\theta} \,, \quad \frac{\Sigma_3}{\sqrt[4]{\Sigma_4^3}} = 2M_2 e^{-\theta} \,, \quad P^2 = (l + \frac{1}{2})^2 - \omega^2 \Sigma_2 \,. \tag{15}$$

the ODE takes the form

$$-\frac{d^2}{dy^2}\psi + \left[e^{2\theta}(e^{2y} + e^{-2y}) + 2e^\theta (M_1 e^y + M_2 e^{-y}) + P^2\right]\psi = 0 \,. \tag{16}$$

- In this form, this ODE is a generalization of the one for the Perturbed Hairpin Integrable Model:

$$-\frac{d^2}{dy^2}\psi + \left[e^{2\theta}(e^{2y} + e^{-ny}) + 2e^\theta M e^y + P^2\right]\psi = 0 \,. \tag{17}$$

Asymptotic solutions and discrete symmetries

- The regular solutions of (16) at $y \to \pm\infty$ $(j = 1, 2)$ have boundary conditions

$$\begin{aligned} \psi_{-,0}(y) &\simeq 2^{-\frac{1}{2}-M_2} e^{-(\frac{1}{2}+M_2)\theta+(\frac{1}{2}+M_2)y} e^{-e^{\theta-y}}, \quad \mathrm{Re}\, y \to -\infty\,. \\ \psi_{+,0}(y) &\simeq 2^{-\frac{1}{2}-M_1} e^{-(\frac{1}{2}+M_1)\theta-(\frac{1}{2}+M_1)y} e^{-e^{\theta+y}}, \quad \mathrm{Re}\, y \to +\infty\,. \end{aligned} \tag{18}$$

- Equation (16) enjoys the discrete symmetries

$$\Omega_\pm : \ \theta \to \theta + i\pi/2\,, \ y \to y \pm i\pi/2\,, \ M_1 \to \mp M_1\,, \ M_2 \to \pm M_2\,, \tag{19}$$

- One can define other independent solutions as

$$\psi_{-,k} = \Omega_-^k \psi_{-,0}\,, \qquad \psi_{+,k} = \Omega_+^k \psi_{+,0}\,. \tag{20}$$

- We also have the invariance properties

$$\Omega_+^k \psi_{-,0} = \psi_{-,0}\,, \qquad \Omega_-^k \psi_{+,0} = \psi_{+,0}\,. \tag{21}$$

Baxter's Q function and quasinormal modes

- One can now define a Baxter's Q function as the wronskian

$$Q_{+,+}(\theta) = W[\psi_{+,0}, \psi_{-,0}] \,. \tag{22}$$

(We will use the notation $Q_{\pm,\pm} = Q(\theta, P, \pm M_1, \pm M_2)$, $Q_{\pm,\mp} = Q(\theta, P, \pm M_1, \mp M_2)$).

- Crucially, the QNMs condition (10) translates into

$$Q(\theta_n) = 0 \,, \tag{23}$$

namely the zeros of the Baxter's Q function which are the Bethe roots.

Central connection relation as QQ system

- The solutions are normalised such that their wronskians are $W[\psi_{-,k+1},\psi_{-,k}] = -i\exp\{(-1)^k i\pi M_2\}$, $W[\psi_{+,k+1},\psi_{+,k}] = i\exp\{(-1)^k i\pi M_1\}$.
- By the properties of wronskians $W[\psi_{\pm,1},\psi_{\pm,1}] = W[\psi_{\pm,0},\psi_{\pm,0}] = 0$, we can write the linear connection relations as

$$
\begin{aligned}
ie^{i\pi M_1}\psi_{-,0} &= Q_{-,+}(\theta + i\pi/2)\psi_{+,0} - Q_{+,+}(\theta)\psi_{+,1} \\
ie^{i\pi M_1}\psi_{-,1} &= Q_{-,-}(\theta + i\pi)\psi_{+,0} - Q_{+,-}(\theta + i\pi/2)\psi_{+,1}
\end{aligned}
\tag{24}
$$

- Taking their wronskian we get the QQ system

$$
Q_{+,-}(\theta + \frac{i\pi}{2})Q_{-,+}(\theta - \frac{i\pi}{2}) = e^{-i\pi(M_1 - M_2)} + Q_{-,-}(\theta)Q_{+,+}(\theta)\,. \tag{25}
$$

Q function as non-compact integral

- From the central connection relation (24) we get a formula for Q

$$Q_{+,+}(\theta) = -ie^{i\pi M_1} \lim_{y\to+\infty} \frac{\psi_{-,0}(y)}{\psi_{+,1}(y)} \tag{26}$$

$$= \int_{-\infty}^{\infty} dy \left[\sqrt{2\cosh(2y)}\Pi(y) - 2e^{\theta}\cosh y - \left(\frac{M_1}{1+e^{-y/2}} + \frac{M_2}{1+e^{y/2}} \right) \right] + \left(\theta + \frac{1}{2}\ln 2 \right)(M_1 - M_2) \tag{27}$$

- The corresponding Riccati equation for $\Pi(y) = \frac{1}{\sqrt{2\cosh(2y)}} \frac{d}{dy} \ln(\sqrt[4]{-2\cosh(2y)}\psi(y))$ is

$$\begin{aligned} &\Pi(y)^2 + \frac{1}{\sqrt{2\cosh(2y)}} \frac{d}{dy}\Pi(y) \\ &= e^{2\theta} + e^{\theta}\operatorname{sech}(2y)\left(M_1 e^{y} + M_2 e^{-y}\right) + \frac{1}{2}\operatorname{sech}(2y)\left[P^2 + \operatorname{sech}(2y) - \frac{5}{4}\tanh^2(2y) \right] \end{aligned} \tag{28}$$

Large energy expansion and Local integrals of motion

- For the $\theta \to +\infty$ expansion, setting

$$\Pi(y,\theta) \doteq e^{\theta} + \sum_{n=0}^{\infty} \Pi_n(y) e^{-n\theta} \qquad \theta \to +\infty \tag{29}$$

we get the recursion relation

$$\Pi_{n+1} = \frac{1}{2}\left(\frac{1}{\sqrt{2\cosh(2y)}} \frac{d}{dy}\Pi_n - \sum_{m=0}^{n} \Pi_m \Pi_{n-m} \right) \qquad n \geq 1 \tag{30}$$

with initial conditions $\Pi_0 = \frac{M_1 e^{y} + M_2 e^{-y}}{2\cosh(2y)}$, $\Pi_1 = \frac{1}{2}\left(\frac{1}{\sqrt{2\cosh(2y)}} \frac{d}{dy}\Pi_0 - \Pi_0^2 - U \right)$.

- The (vaccum eigenvalues) local integrals of motion (LIMs) are given by the integrals

$$\begin{aligned} C_n I_n(P, M_1, M_2) &= \int_{-\infty}^{\infty} dy \, \sqrt{-2\cosh(2y)} \Pi_n(y, P, M_1, M_2) \\ Q(\theta, P, M_1, M_2) &\doteq \exp\left\{ -\frac{4\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2} e^{\theta} + (\dots)\theta + \sum_{n=1}^{\infty} e^{-\theta n} C_n I_n(P, M_1, M_2) \right\} \qquad \theta \to +\infty \end{aligned} \tag{31}$$

Y function and quasinormal modes

- One can define a Y function as

$$Y_{+,+}(\theta) = e^{i\pi(M_1-M_2)} Q_{+,+}(\theta) Q_{-,-}(\theta) \tag{32}$$

- From the QQ system it follows the Y system as

$$Y_{+,-}(\theta + \frac{i\pi}{2}) Y_{-,+}(\theta - \frac{i\pi}{2}) = [1 + Y_{+,+}(\theta)][1 + Y_{-,-}(\theta)] \,. \tag{33}$$

- Eventually, the QQ system written as

$$e^{i\pi(M_1+M_2)} Q_{+,+}(\theta + \frac{i\pi}{2}) Q_{-,-}(\theta - \frac{i\pi}{2}) = 1 + Y_{+,-}(\theta) \,. \tag{34}$$

characterizes the QNMs as

$$Y_{+,-}(\theta_n - i\pi/2) = -1 \qquad Y_{-,+}(\theta_n + i\pi/2) = -1 \tag{35}$$

Thermodynamic Bethe Ansatz

- It can be inverted in a Thermodynamic Bethe Ansatz (TBA) for $\varepsilon_{\pm,\pm}(\theta) = -\ln Y_{\pm,\pm}(\theta)$:

$$\varepsilon_{\pm,\pm}(\theta) = \frac{8\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2} e^\theta \mp i\pi(M_1 - M_2) - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{\ln[1+\exp(-\varepsilon_{\pm,\mp}(\theta'))] + \ln[1+\exp(-\varepsilon_{\mp,\pm}(\theta'))]}{\cosh(\theta-\theta')}$$
$$\varepsilon_{\pm,\mp}(\theta) = \frac{8\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2} e^\theta \mp i\pi(M_1 + M_2) - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{\ln[1+\exp(-\varepsilon_{\pm,\pm}(\theta'))] + \ln[1+\exp(-\varepsilon_{\mp,\mp}(\theta'))]}{\cosh(\theta-\theta')} \tag{36}$$

- P enters as the $\theta \to -\infty$ boundary condition (from perturbative expansion of ODE)

$$\varepsilon_{+,+}(\theta, P) \sim 4P\theta + 2C(P, M_1, M_2)\,, \qquad \theta \to -\infty \tag{37}$$

$$C(P, M_1, M_2) = \ln\left[\frac{2^{1-2P} P\, \Gamma(2P)^2}{\sqrt{\Gamma\left(P+\frac{1}{2}-M_1\right)\Gamma\left(P+\frac{1}{2}+M_1\right)\Gamma\left(P+\frac{1}{2}-M_2\right)\Gamma\left(P+\frac{1}{2}-M_2\right)}}\right] \tag{38}$$

Gravity Y system and TBA

- In gravity variables the Y system reads

$$
\begin{aligned}
&Y(\theta+\frac{i\pi}{2},-i\Sigma_1,-\Sigma_2,i\Sigma_3,\Sigma_4)Y(\theta-\frac{i\pi}{2},-i\Sigma_1,-\Sigma_2,i\Sigma_3,\Sigma_4)\\
&=[1+Y(\theta,\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4)][1+Y(\theta,-\Sigma_1,\Sigma_2,-\Sigma_3,\Sigma_4)]
\end{aligned} \tag{39}
$$

- It can be inverted into the TBA

$$
\begin{aligned}
\varepsilon_{\pm,\pm}(\theta)&=[f_{0,+}\mp\frac{i\pi}{2}(\frac{\Sigma_1}{\Sigma_4^{1/4}}-\frac{\Sigma_3}{\Sigma_4^{3/4}})]e^{\theta}-\varphi*(\bar{L}_{\pm\pm}+\bar{L}_{\mp\mp})(\theta)\\
\bar{\varepsilon}_{\pm,\pm}(\theta)&=[\bar{f}_{0,+}\pm\frac{\pi}{2}(\frac{\Sigma_1}{\Sigma_4^{1/4}}+\frac{\Sigma_3}{\Sigma_4^{3/4}})]e^{\theta}-\varphi*(L_{\pm\pm}+L_{\mp\mp})(\theta)
\end{aligned} \tag{40}
$$

where we defined $\varepsilon(\theta)=-\ln Y(\theta,\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4)$, $\bar{\varepsilon}(\theta)=\varepsilon(\theta,i\Sigma_1,-\Sigma_2,-i\Sigma_3,\Sigma_4)$, $L=\ln[1+\exp\{-\varepsilon\}]$, $\varphi(\theta)=(2\pi\cosh(\theta))^{-1}$, $f_{0,+}=c_{0,+,+}+c_{0,-,-}$ and $c_{0,\pm,\pm}=\int_{-\infty}^{\infty}\left[\sqrt{2\cosh(2y)\pm\frac{\Sigma_1}{\sqrt[4]{\Sigma_4}}e^{y}\pm\frac{\Sigma_3}{\sqrt[4]{\Sigma_4^3}}e^{-y}+\frac{\Sigma_2}{\sqrt{\Sigma_4}}}-\text{reg.}\right]dy$ and boundary condition $\varepsilon_{\pm,\pm}(\theta)\sim 4(l+\frac{1}{2})\theta+2C(P,0,0)$, $\theta\to-\infty$.

Quasinormal modes from TBA

- The QNMs condition in gravity variables reads alternatively as

$$\bar{Y}_{+,+}(\theta_{n'} - i\pi/2) = -1\,, \quad Q_{+,+}(\theta_n) = 0 \tag{41}$$

or

$$\bar{\varepsilon}_{+,+}(\theta_{n'} - i\pi/2) = -i\pi(2n' + 1)\,. \tag{42}$$

- Through the quantization condition on $\bar{\varepsilon}_{+,+}$ we can actually numerically compute QNMs from TBA.

n	l	TBA	Leaver	WKB
0	1	$\underline{0.869}623 - \underline{0.372}022i$	$\underline{0.869}932 - \underline{0.372}859i$	$0.89642 - 0.36596i$
0	2	$\underline{1.477}990 - \underline{0.368}144i$	$\underline{1.477}888 - \underline{0.368}240i$	$1.4940 - 0.36596i$
0	3	$\underline{2.080}200 - \underline{0.367}076i$	$\underline{2.080}168 - \underline{0.367}097i$	$2.0916 - 0.36596i$
0	4	$\underline{2.680}363 - \underline{0.366}637i$	$\underline{2.680}350 - \underline{0.366}642i$	$2.6893 - 0.36596i$

Table: Comparison of QNMs in different methods ($n' = 0$, $\Sigma_1 = \Sigma_3 = 0.2$, $\Sigma_2 = 0.4$, $\Sigma_4 = 1$).

Explanation of connection to $\mathcal{N} = 2$ gauge theory

Classical Seiberg-Witten curve and differential

- Given an operator $H(\hat{x}, \hat{p})$, with $[\hat{x}, \hat{p}] = i\hbar$, we associate it with a classical curve and a one form

$$H(x, p) = u \,, \qquad \lambda(x, u) = f(p(x, u))\, d\, x \tag{43}$$

- We consider operators $H(\hat{x}, \hat{p})$ such that the corresponding classical curve (43) coincides with a SW curve of a suitable 4D $\mathcal{N} = 2$ gauge theory.
- In particular we will consider $SU(2)$ $N_f = 2$ gauge theory, with SW curve and SW differential:

$$p^2 = x^3 - x^2 u - \frac{\Lambda_2^4}{64}(x - u) + \frac{\Lambda_2^2}{4} m_1 m_2 x - \frac{\Lambda_2^4}{64}(m_1^2 + m_2^2) \,. \tag{44}$$

$$\lambda = -\frac{\sqrt{2}}{4\pi}\frac{dx}{p}\left[x - u - \frac{\Lambda_2^2}{16}\frac{(m_1 - m_2)^2}{x - \frac{\Lambda_2^2}{8}} + \frac{\Lambda_2^2}{16}\frac{(m_1 + m_2)^2}{x + \frac{\Lambda_2^2}{8}}\right] . \tag{45}$$

Classical Seiberg-Witten periods

- If the curve (43) has genus g, we choose a basis of cycles $\mathcal{A}_i, \mathcal{B}_i$, with $i = 1, \ldots, g$ and define classical periods by integrating over such cycles

$$\Pi_{A_i,B_i}(u) = \oint_{\mathcal{A}_i,\mathcal{B}_i} \lambda(x,u)\, dx\,, \qquad i = 1, \ldots, g\,. \tag{46}$$

- (46) are the SW periods and they encode the central charges and masses of the BPS particles in the theory.
- For $SU(2)$ $N_f = 2$:

$$\int_{A,B} \lambda = \frac{\sqrt{2}}{4\pi}\left[\frac{4}{3}uI_1^{A,B} - 2I_2^{A,B} + \frac{\Lambda_2^2}{8}(m_1-m_2)^2 I_3^{A,B}\left(\frac{\Lambda_2^2}{8} - \frac{u}{3}\right) - \frac{\Lambda_2^2}{8}(m_1+m_2)^2 I_3^{A,B}\left(-\frac{\Lambda_2^2}{8} - \frac{u}{3}\right)\right] \tag{47}$$

The integral is taken over different cycles A and B between the branch points e_1, e_2, e_3 of the SW curve p. I_1, I_2, I_3 are combinations of elliptic integrals of 1st, 2nd and 3rd kind.

Quantum Seiberg-Witten curve and integrability dictionary

- Quantum Seiberg Witten curve in the Nekrasov-Shatashvili limit of the Ω-background (used to compute instanton contributions) $\epsilon_2 \to 0, \epsilon_1 = \hbar$ with $x \sim e^{-y}$

$$H_{N_f}\psi(y) = u\psi(y) \tag{48}$$

$$H_{N_f} = -\hbar^2\frac{d^2}{dy^2} + \frac{1}{2}\Lambda_{N_f}^{2-N_f/2}\left[e^y K_+\left(\hbar\frac{d}{dy} - u_1 + \frac{\hbar}{2}\right) + e^{-y}K_-\left(\hbar\frac{d}{dy} - u_1 - \frac{\hbar}{2}\right)\right] - u_0 \ . \tag{49}$$

- For $N_f = 2 = (1,1)$ we have

$$-\hbar^2\frac{d^2}{dy^2}\psi + \left[\frac{\Lambda_2^2}{8}\cosh(2y) + \frac{1}{2}m_1\Lambda_2 e^y + \frac{1}{2}m_2\Lambda_2 e^{-y} + u\right] \tag{50}$$

- We get this equation from the Generalized Perturbed Hairpin Integrable model ODE (16) through the dictionary

$$\frac{\hbar}{\Lambda_2} = \frac{1}{4}e^{-\theta}, \qquad \frac{u}{\Lambda_2^2} = \frac{1}{16}P^2e^{-2\theta}, \qquad \frac{m_1}{\Lambda_2} = \frac{1}{4}M_1e^{-\theta}, \qquad \frac{m_2}{\Lambda_2} = \frac{1}{4}M_2e^{-\theta}, \tag{51}$$

Quantum SW periods

- We can define exact periods by exact integrals of $\mathcal{P}(y) = -i\frac{d}{dy}\ln\psi(y)$ as sums over residues at the poles which as $\hbar \to 0$ reduce to the classical cycles (branch cuts).

$$\Pi_{A,B}(u,\hbar) \doteq \oint_{A,B} \mathcal{P}(\hbar,y,u)\,dy = 2\pi i \sum_n \mathrm{Res}\mathcal{P}(y)\Big|_{y_n^{A,B}} \qquad \Pi_A = a\,, \quad \Pi_B = a_D \tag{52}$$

- Alternatively, we can define
  - the B period through the Nekrasov-Shatashvili free energy $\mathcal{F}^{(N_f)}(a,\mathbf{m},\Lambda_{N_f},\hbar)$, which can be computed systematically through combinatorial calculus on the gauge group representation:

$$A_D(u,\mathbf{m},\Lambda_{N_f},\hbar) = \Pi_B^{inst}(u,\mathbf{m},\Lambda_{N_f},\hbar) = \partial_a\mathcal{F}(u,\mathbf{m},\Lambda_{N_f},\hbar)\Big|_{a=a(u,\mathbf{m},\Lambda_{N_f},\hbar)} \tag{53}$$

  - the A period $\Pi_A = a$ from Matone's relation:

$$u = a^2 - \frac{\Lambda_{N_f}}{4-N_f}\frac{\partial\mathcal{F}^{(N_f)}_{\text{inst}}(a;\mathbf{m},\Lambda_{N_f},\hbar)}{\partial\Lambda_{N_f}} \tag{54}$$

Gauge TBA

- In gauge variables the Y system reads

$$Y(\theta+\frac{i\pi}{2},-u,-im_1,im_2,\Lambda_2)Y(\theta-\frac{i\pi}{2},-u,-im_1,im_2,\Lambda_2) \\ =[1+Y(\theta,u,m_1,m_2,\Lambda_2)][1+Y(\theta,u,-m_1,-m_2,\Lambda_2)] \tag{55}$$

so it can be inverted in the TBA similar to but different from the gravity one (56)

$$\varepsilon_{\pm,\pm}(\theta)=[f_{0,+}\mp\frac{4\pi i}{\Lambda_2}(m_1-m_2)]e^{\theta}-\varphi*(\bar{L}_{\pm\pm}+\bar{L}_{\mp\mp})(\theta) \\ \bar{\varepsilon}_{\pm,\pm}(\theta)=[\bar{f}_{0,+}\mp\frac{4\pi}{\Lambda_2}(m_1+m_2)]e^{\theta}-\varphi*(L_{\pm\pm}+L_{\mp\mp})(\theta) \tag{56}$$

where we defined $\varepsilon(\theta)=-\ln Y(\theta,u,m_1,m_2,\Lambda_2)$, $L=\ln[1+\exp\{-\varepsilon\}]$ and in particular $\bar{\varepsilon}(\theta)=\varepsilon(\theta,-u,-im_1,+im_2,\Lambda_2)$. The kernel is still $\varphi(\theta)=\frac{1}{2\pi}\frac{1}{\cosh\theta}$. The forcing term is given by $f_{0,\pm}=-c_{0,+,\pm}-c_{0,-,\mp}$ with $c_{0,\pm,\pm}=\int_{-\infty}^{\infty}\left[\sqrt{2\cosh(2y)\pm\frac{8m_1}{\Lambda_2}e^{y}\pm\frac{8m_2}{\Lambda_2}e^{-y}+\frac{16u}{\Lambda_2^2}}-\text{reg.}\right]dy$.

Quantum gauge B period from TBA

- We find that the $\theta \to \infty$ asymptotic expansion for $\varepsilon = \sum_{n=0}^{\infty} e^{\theta(1-2n)} \varepsilon^{(n)}$ gives actually the gauge periods $\hbar \to 0$ asymptotic expansion $a_D \doteq \sum_{n=0}^{\infty} \hbar^{2n} a_D^{(n)}$ coefficients as computed from the ODE (WKB approximation).

$$
\begin{aligned}
\bar{\varepsilon}_{+,+}^{(n)} &= \varepsilon^{(n)}(-u, im_1, -im_2, \Lambda_2) = (-1)^n \frac{8\sqrt{2}\pi}{\Lambda_2} a_D^{(n)}(u, m_1, m_2, \Lambda_2) \\
\bar{\varepsilon}_{-,-}^{(n)} &= \varepsilon^{(n)}(-u, -im_1, im_2, \Lambda_2) = (-1)^n \frac{8\sqrt{2}\pi}{\Lambda_2} a_D^{(n)}(u, m_1, m_2, \Lambda_2) + \delta_{n,0} \frac{8\pi}{\Lambda_2}(m_1 + m_2)
\end{aligned} \tag{57}
$$

- Also considering the exact integrals we can prove

$$
\begin{aligned}
\varepsilon(\theta, -u, im_1, -im_2, \Lambda_2) &= \frac{8\sqrt{2}\pi}{\Lambda_2} a_D(\hbar, u, m_1, m_2, \Lambda_2) \\
\varepsilon(\theta, -u, -im_1, im_2, \Lambda_2) &= \frac{8\sqrt{2}\pi}{\Lambda_2} a_D(\hbar, u, m_1, m_2, \Lambda_2) + \frac{8\pi}{\Lambda_2}(m_1 + m_2) e^{\theta}
\end{aligned} \tag{58}
$$

Quantum gauge B period as pseudoenergy: geometrical proof

- We can prove analytically the equality of the pseudonergy and gauge period by relating the different integration contours in the complex plane.

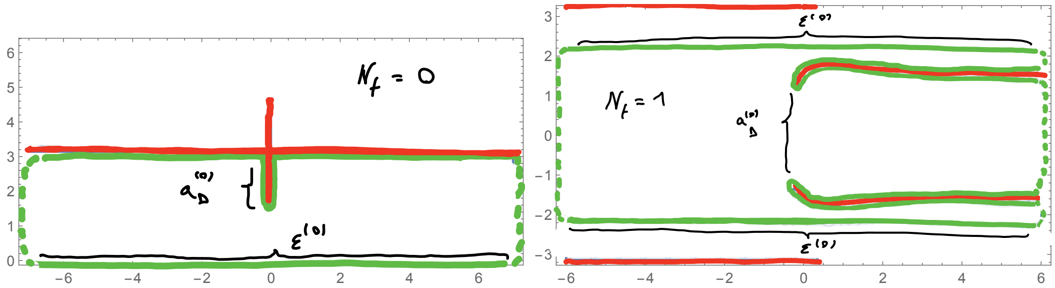


Figure: Proof of the equality of pseudonergy $\varepsilon^{(0)}$ and gauge dual period $a_D^{(0)}$ at the classical order, for the $SU(2)$ $N_f = 0, 1$ gauge theories. In red the branch cuts, in green the integration contour.

Quasinormal modes from gauge periods quantization conditions

- We see that from our formalism it follows immediately that QNMs are given by quantization conditions on the gauge integral periods

$$\frac{8\sqrt{2}\pi}{\Lambda_2} a_D(\hbar, u, m_1, m_2, \Lambda_2) = -i\pi(2n' + 1) \tag{59}$$

- Recently other authors found QNMs to be given by quantization conditions on the gauge instanton periods

$$A_D(\hbar, u, m_1, m_2, \Lambda_2) = i\pi n \tag{60}$$

- In the limit to $N_f = 0$ $SU(2)$ we know a consistent relation between them

$$Q(\theta, P) = \exp\left\{\frac{2\pi i}{\hbar} a_D(u, \Lambda_0, \hbar)\right\} = i\frac{\sinh \frac{1}{\hbar} A_D(a, \Lambda_0, \hbar)}{\sinh \frac{2\pi i}{\hbar} a}, \tag{61}$$

Baxter's T functions and TQ relations

- Now, the presence of the irregular singularity of (16) at $y \to \pm\infty$ (Stokes phenomenon) plays a rôle for defining the T functions

$$T_{+,+}(\theta) = -iW[\psi_{-,-1}, \psi_{-,1}]\,, \quad \tilde{T}_{+,+}(\theta) = iW[\psi_{+,-1}, \psi_{+,1}]\,. \tag{62}$$

By expanding $\psi_{\pm,1}$ in terms of $\psi_{\pm,0}$, $\psi_{\pm,-1}$ as

$$\psi_{+,-1} = -e^{-2\pi i M_1}\psi_{+,1} + e^{-i\pi M_1}\tilde{T}_{+,+}(\theta)\psi_{+,0} \tag{63}$$

$$\tilde{T}_{+,+}(\theta)\psi_{+,0} = e^{i\pi M_1}\psi_{+,-1} + e^{-i\pi M_1}\psi_{+,+1} \tag{64}$$

we obtain the TQ relations

$$\begin{aligned} T_{+,+}(\theta)Q_{+,+}(\theta) &= e^{i\pi M_2}Q_{+,-}(\theta - \frac{i\pi}{2}) + e^{-i\pi M_2}Q_{+,-}(\theta + \frac{i\pi}{2}) \\ \tilde{T}_{+,+}(\theta)Q_{+,+}(\theta) &= e^{i\pi M_1}Q_{-,+}(\theta - \frac{i\pi}{2}) + e^{-i\pi M_1}Q_{-,+}(\theta + \frac{i\pi}{2})\,. \end{aligned} \tag{65}$$

- By the invariance under the symmetries $\Omega_\pm$ also the periodicities follow

$$T_{-,+}(\theta + i\pi/2) = T_{+,+}(\theta)\,, \quad \tilde{T}_{+,-}(\theta + i\pi/2) = \tilde{T}_{+,+}(\theta) \tag{66}$$

Baxter's T function, Floquet exponent and gauge A period

- One can prove a relation between the T function for the doubly confluent Heun equation and its Floquet exponent (such that $\psi(y + 2\pi i) = e^{2\pi i\nu}\psi(y)$)

$$\begin{aligned} T_{+,+}(\theta)\, T_{+,-}(\theta + i\pi/2) &= 4\cos(\pi\nu + \pi m_1)\cos(\pi\nu - \pi m_1) \\ \tilde{T}_{+,+}(\theta)\, \tilde{T}_{-,+}(\theta + i\pi/2) &= 4\cos(\pi\nu + \pi m_2)\cos(\pi\nu - \pi m_2) \end{aligned} \tag{67}$$

- For the $SU(2)$ $N_f = 0$ gauge theory by definition

$$\nu = \Pi_A = a \tag{68}$$

and therefore another fundamental link between integrability and gauge theory:

$$T(\theta, P) = \tilde{T}(\theta, P) = 2\cos 2\pi a(u, \Lambda_0, \hbar) \tag{69}$$

- For $SU(2)$ $N_f = 2$ we are in the process of proving an analogue relation between ν and a.

Proof of alternative quantization condition for quasinormal modes

- From the relation between T and a it follows a proof of the alternative quantization condition for quasinormal modes

$$a(\theta_n) = \frac{n}{2}\,, \qquad n \in \mathbb{Z}\,. \tag{70}$$

which was given by some authors for the D3 brane (corresponding to $SU(2)$ $N_f = 0$).

- Indeed, the TQ relation $T(\theta)Q(\theta) = Q(\theta - i\pi/2) + Q(\theta + i\pi/2)$ means $Q(\theta_n - i\pi/2) + Q(\theta_n + i\pi/2) = 0$. This and the QQ relation $Q(\theta + i\pi/2)Q(\theta - i\pi/2) = 1 + Q^2(\theta)$ actually fixes $Q(\theta_n + i\pi/2)Q(\theta_n - i\pi/2) = 1$ and then

$$Q(\theta_n \pm i\pi/2) = \pm i \tag{71}$$

are fixed, too. Again the QQ relation around θ_n forces $Q(\theta + i\pi/2) = i \pm Q(\theta) + \dots$ and $Q(\theta - i\pi/2) = -i \pm Q(\theta) + \dots$ up to smaller corrections (dots). Therefore, (**??**) imposes

$$T(\theta_n) = 2\cos 2\pi a(\theta_n) = \pm 2\,. \tag{72}$$

Partial generalization of alternative quantization condition for quasinormal modes

- By making considerations on these TQ systems and the QQ system (34) like in section III, we are not in general able to conclude that the T-s are quantized, except in the case of equal masses $M_1 = M_2$ where we find

$$T_{+,+}(\theta_n)T_{-,-}(\theta_n) = 4\,, \qquad (M_1 = M_2)\,. \tag{73}$$

that generalizes (72). On the other hand, in this case we have the symmetry

$$\psi_{+,0}(y) = \psi_{-,0}(-y)\,, \qquad (M_1 = M_2) \tag{74}$$

and, since $\psi_{\pm,0}$ are Floquet solutions ($\psi_{\pm,0}(y+2\pi i) = e^{\pm 2\pi i\nu}\psi_{\pm,0}(y)$), the QNMs condition (10) requires the Floquet index ν to be quantized. Hence we conjecture a relation between T, ν and a like for $N_f = 0$, that would generalize the alternative quantization condition (70) only if $M_1 = M_2$.

Interpretation of integrability duality

- Moreover $\psi_{+,0}(y) = \psi_{-,0}(-y)$ is just the symmetry that exchange infinity ($y \to +\infty$) and the (analogue) horizon ($y \to -\infty$), leaving the photon sphere ($y = 0$) fixed as in [BianchiRusso:2021] (thanks to identifications of certain scattering angles with the SW period). In this respect, under this symmetry we have

$$\tilde{T}_{+,+}(\theta) = T_{+,+}(\theta) \qquad (M_1 = M_2) \tag{75}$$

would be consistent with a generalization of $T(\theta, P) = 2\cos 2\pi a(u, \Lambda_0, \hbar)$, which happens at the Liouville self-dual point $b = 1$ where (74) holds.

Comparison of methods of computation of quasinormal modes

Comparison of methods of computation of quasinormal modes

n	l	TBA	Leaver	WKB
0	1	$0.896681 - 0.40069i$	N.A.	$0.93069 - 0.39458i$
0	2	$1.5308 - 0.39676i$	N.A.	$1.5511 - 0.39458i$
0	3	$2.15708 - 0.395689i$	N.A.	$2.1716 - 0.39458i$
0	4	$2.78077 - 0.39525i$	N.A.	$2.7921 - 0.39458i$

Table: Here $\Sigma_1 \neq \Sigma_3$ and the Leaver method seems not applicable (N.A.).

- Computing QNMs numerically has been until now typically quite laborious and this has been also due to the difficulties in developing exact analytic characterizations of QNMs.
- The standard analytic method is the one with the continuous fractions by Leaver. However, it requires that the ODE is such that its solution be expressed in power series with coefficients satisfying a 3-point recursion.
- A new analytic characterization appeared very recently as QNMs have been obtained by Nekrasov partition functions of $\mathcal{N}=2$ supersymmetric gauge, *i.e.* theories. Nevertheless, we found it very difficult to reach, by summing instantons, the QNMs values $|\Lambda_{N_f}|/\hbar \gg 1$. It needs a clever numerical re-summation procedure.

Generalization and perspectives

Present limitations and future generalizations

- What we have described holds for the generalizazion of RN BHs (intersection of 4 stacks of D3 branes). It corresponds to $SU(2)$ $N_f = 2, 1, 0$ gauge theories.
- However, from the generality of construction it is manifest that our method applies to many other theories. We can list, for instance, the $SU(2)$ $N_f = (2, 0)$ gauge theory and the associated gravity counterparts, like D1D5 fuzzballs and CCLP 5D BHs, as well as to $SU(2)$ $N_f = 3$ gauge theory and the general asymptotically flat Kerr-Newman BH. The associated integrable structure appears to be a generalization of the conformal minimal models. In these cases the (confluent Heun) ODE has one irregular and two regular singulatities.
- Another simple but tricky case happens when all the (Heun) ODE singularities are regular, e.g. $SU(2)$ $N_f = 4$ or $SU(2)$ quivers that is asymptotically AdS BHs. The corresponding IM appear to be spin chains.
- Analytic continuations of the TBAs in θ and the moduli are necessary to obtain overtones ω_n $n \geq 1$ and some particular gravitational systems.

Perspectives

- Besides generalization, it is very intriguing to study the application of the integrability structure beyond the determination of the QNMs for the study of gravitational solutions.
- For AdS BHs the 3-fold correspondence Integrability/Gravity/Gravity could become 4-fold through holographic duality to CFT. The so called "poles skipping phenomenon" for the retarded Green function of the holographic CFT is currently being investigated.

Great thanks for all and goodbye,
dear NORDITA!