

# From Integrability to Supersymmetry and Black Holes

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Strings, Gravity and Gravitational Waves,

Rizhao, August 2<sup>nd</sup> 2024

Based on: [arXiv:24\\*\\*.\\*\\*\\*\\*\\*](#), [arXiv:2208.14031](#), [arXiv:2112.11434](#), [arXiv:1908.08030](#)

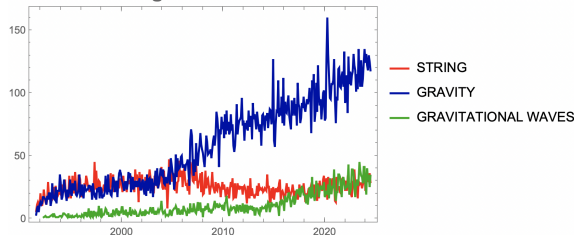
with D. Fioravanti, R. Mahanta, M. Rossi, H. Shu

# Introduction

# Strings, Gravity & Gravitational Waves

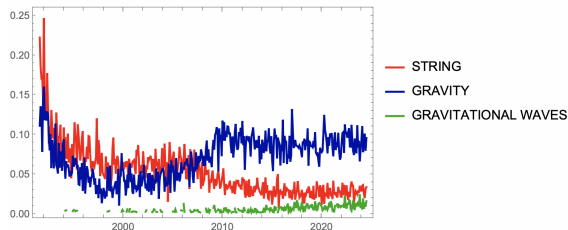
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Total new articles on all ArXiv  
with a given a title word



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Word share trends in hep-th titles



To draw such plots\* just use my **ArXivExplore** paclet!

\*as well as for computing **all-ArXiv** title/abstract **word statistics**; TeX source, **formulae**, **citations** extraction; **Neural Networks** for classification/recommendation; **LLM** concept **explanations** & **author reports**.

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# GWs: “A new window to the universe”

In the last few years, **gravitational waves (GWs)** detections have opened the way for **gravitational phenomenology**.

These new observations promise to revolutionise even **fundamental physics**, allowing the search of **deviations from general relativity (GR)** in previously unexplored regimes.

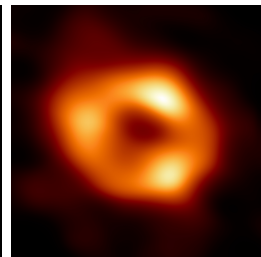
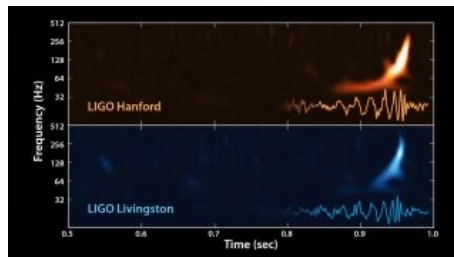


Figure: LIGO, Event Horizon

Since **BHs are the simplest and most perfect objects** in gravity, one may ask

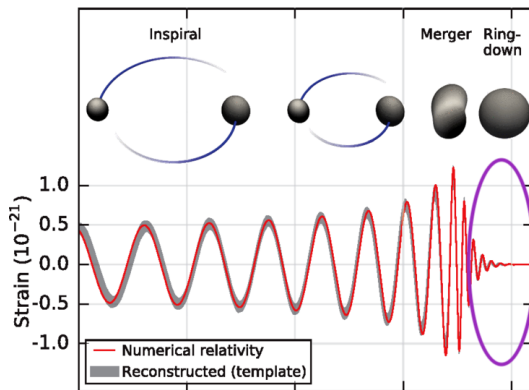
*Could BHs hold the same surprises that the electron and the hydrogen atom did when they started to be experimentally probed?* [[Cardoso-Pani, arXiv:1709.01525](#)]

# Colliding BHs and quasinormal modes

A **black hole collision** can be divided in 3 phases: **inspiral**, **merger** and **ringdown**.

QNMs can be observed as the complex frequencies of the **damped oscillations of spacetime** in the final - **ringdown** - phase of black holes (BHs) merging.

[Nollert, Class.Quant.Grav. 16 (1999);  
Berti-Cardoso-Starinets, arXiv:0905.2975]



Black holes perturbations can be expressed in terms of solutions of **ODEs shared with other physics fields!**

# The need of new methods

- Even if QNMs for GR BHs are already known, it is still an open problem to compute them in **modified gravity**. Specifically, the common methods are:
  - Continued fraction** method: exact but laborious;
  - WKB approximation**: easy but rough, asymptotic, to be resummed;
  - Numeric simulations**: case by case, sometimes unstable, with little insight.

Therefore developing **new analytic characterizations and exact numerical methods** for QNMs is very important.

- QNMs also present **theoretical problems even in GR** like their instability under small perturbations of the background.

$$\frac{1}{4x^2(x-1)^3} \sum_{n=0}^2 \sum_{m=0}^{2-n} A_{n,m} g^{(n)}(x) (i\Omega)^m = 0. \quad (21)$$

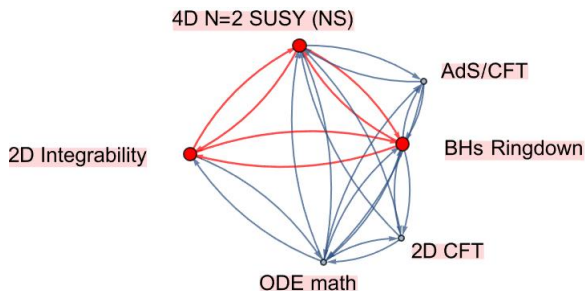
Here  $g^{(n)}(x)$  is the  $n$ -th derivative of the function  $g(x)$ . The real coefficients  $A_{n,m}(x)$  are

$$\begin{aligned} A_{0,0} &= -x^2(x-1)^2 [4(\ell-1)(\ell+2)\mathcal{M}_0\mathcal{S}_0^3 \\ &\quad - x(8\mathcal{N}_0\mathcal{S}_0' + x\mathcal{N}_0'\mathcal{S}_0' + 2x\mathcal{N}_0\mathcal{S}_0'')\mathcal{M}_0\mathcal{W}_0\mathcal{S}_0 \\ &\quad + x(2\mathcal{S}_0 - x\mathcal{S}_0')\mathcal{M}_0'\mathcal{N}_0\mathcal{W}_0\mathcal{S}_0 \\ &\quad + 3x^2\mathcal{M}_0\mathcal{N}_0\mathcal{W}_0\mathcal{S}_0'^2 \\ &\quad + 2(4\mathcal{N}_0 + x\mathcal{N}_0')\mathcal{M}_0\mathcal{W}_0\mathcal{S}_0^2] \mathcal{P}_0 \\ A_{0,1} &= -2x^2 [2(\delta + (x^2 - 2x)(\gamma + \delta))\mathcal{M}_0\mathcal{N}_0 \\ &\quad + (x-1)B(x)(\mathcal{M}_0\mathcal{N}_0)'] \mathcal{P}_0\mathcal{W}_0\mathcal{S}_0^2, \\ A_{0,2} &= +4[B^2(x)\mathcal{M}_0\mathcal{N}_0\mathcal{P}_0 - (x-1)^2\mathcal{S}_0]\mathcal{W}_0\mathcal{S}_0^2, \\ A_{1,0} &= +2x^3(x-1)^2 [x(\mathcal{M}_0\mathcal{N}_0)' + 4\mathcal{M}_0\mathcal{N}_0] \mathcal{P}_0\mathcal{W}_0\mathcal{S}_0^2, \\ A_{1,1} &= -8x^2(x-1)B(x)\mathcal{M}_0\mathcal{N}_0\mathcal{P}_0\mathcal{W}_0\mathcal{S}_0^2, \\ A_{2,0} &= +4x^4(x-1)^2\mathcal{M}_0\mathcal{N}_0\mathcal{P}_0\mathcal{W}_0\mathcal{S}_0^2, \end{aligned}$$

Figure: ODE for axial perturbation in Einstein scalar Gauss-Bonnet theory [[arXiv:2404.11583](#)]

# Plan of the talk

- 1 Introduction
- 2 SW theory and BHs
- 3 ODE/IM correspondence and BHs
  - Introduction to ODE/IM correspondence
  - QNMs as Bethe roots
  - Thermodynamics Bethe Ansatz
  - Solving quantum Wronskians of gauge theories
- 4 Conclusions



# SW theory and BHs

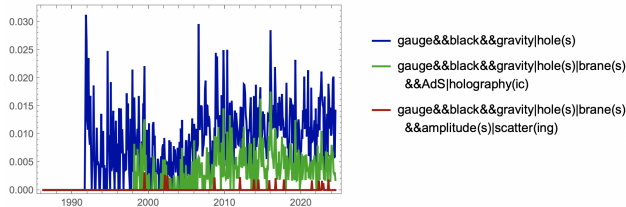


# To the rescue of supersymmetry?

We see an increasing of article abstracts talking simultaneously of **gauge theory and black holes**.

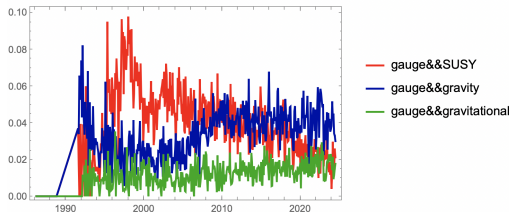
Actually, it is mostly due to **AdS/CFT** and NOT to the well known community of **scattering amplitudes**, modeling the inspiral phase of black holes as particle collision.

Combination share trends in hep-th abstracts



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Combination share trends in hep-th abstracts

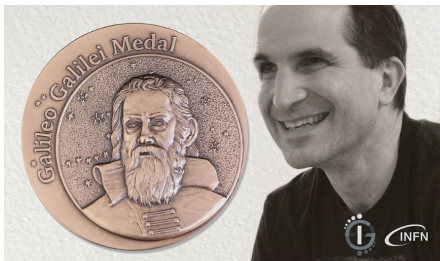


An even smaller community works differently with **Seiberg-Witten theory** on (mostly) the ringdown phase...

Are AdS black holes more **physical** than supersymmetric particles?

# Galileo and Maldacena

At least twice in my short scientific life - during formal meetings at Bologna University - **I have been publicly attacked for proposing the study of ( $\mathcal{N} = 2$ ) supersymmetry...**



Similarly - according to rumours - someone complained for the Galileo Galilei award to Prof. Maldacena, on the basis of “**not being an experimental physicist, like Galileo was**” ...

...but in the *Saggiatore* we read also “the book of nature is written in...”

# “The book of nature is written in mathematical language!”

Generally in both **General Relativity** and **modified gravity**, BHs perturbations obey different kind of ODEs.

- Perturbations of **asymptotically AdS<sub>5</sub> BHs** obey **Heun** equation

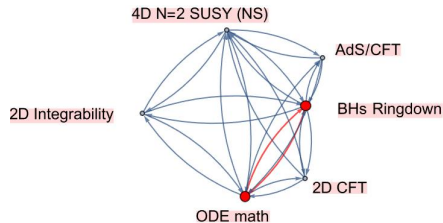
$$w''(z) + \left( \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\alpha + \beta - \gamma - \delta + 1}{z-a} \right) w'(z) + \frac{\alpha\beta z - q}{z(z-1)(z-a)} w(z) = 0. \quad (1)$$

- for 4D **asymptotically flat BHs**, **Confluent Heun** equation (CHE)

$$w''(z) + \left( \frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon \right) w'(z) + \frac{\alpha z - q}{z(z-1)} w(z) = 0. \quad (2)$$

- for 4D **Extremal** flat BHs, **Doubly Confluent Heun** eq. (DCHE)

$$w''(z) + \left( \frac{\delta}{z^2} + \frac{\gamma}{z} + 1 \right) w'(z) + \frac{\alpha z - q}{z^2} w(z) = 0. \quad (3)$$



- Caveat: **rotating BHs** involve **coupled ODEs** (separable PDEs)!
- The inspiral phase is governed by the **non-homogenous** same ODEs!

# $\mathcal{N} = 2$ SUSY primer

- **Seiberg-Witten (SW) theory** is the effective theory for  $\mathcal{N} = 2$  SUSY gauge theory. To compute **instanton contributions**, it is convenient to deform spacetime in terms of two parameters  $\epsilon_1, \epsilon_2 \in \mathbb{C}$  ( **$\Omega$ -background**).  
[Seiberg-Witten, hep-th/9407087; Nekrasov, hep-th/0206161]
- If  $\epsilon_2 \rightarrow 0$  - **Nekrasov-Shatashvili (NS) limit** - the prepotential  $\mathcal{F}$  can be obtained from monodromies of ODEs of Schrödinger time-independent type with  $\epsilon_1 = \hbar \neq 0$  (for  $SU(2)$ )

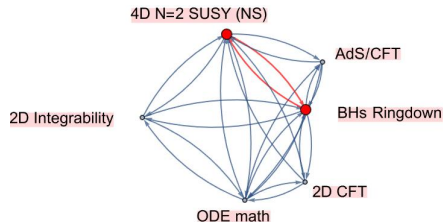
$$-\hbar^2 \frac{d^2}{dy^2} \psi(y) + [U(y; \mathbf{m}, \Lambda) - u] \psi(y) = 0, \quad (4)$$

[Nekrasov-Shatashvili, arXiv:0908.4052]

# No jokes: Schwarshild black hole!

The quantum SW curve for  $N_f = 3$   $SU(2)$  **gauge th.** can be mapped to the Regge-Wheeler equation for linear **perturbations of the Schwarshild BH!**

$$\begin{aligned}
 & -\hbar^2 \frac{d^2}{dy^2} \psi(y) + \left\{ e^{2y} \Lambda_3 (4(m_1 - m_2)^2) \right. \\
 & + (\Lambda_3^2 - 24\Lambda_3 m_3 + 64u) + 4e^y \sqrt{\Lambda_3} (-2\hbar^2 + 8m_1 m_2 + \Lambda_3 m_3 - 8u) \\
 & \left. + 4e^{-y} \sqrt{\Lambda_3} (8m_3 - \Lambda_3) + 4\Lambda_3 e^{-2y} \right\} \frac{1}{16 (\sqrt{\Lambda_3} e^y - 2)^2} \psi(y) = 0
 \end{aligned} \tag{5}$$

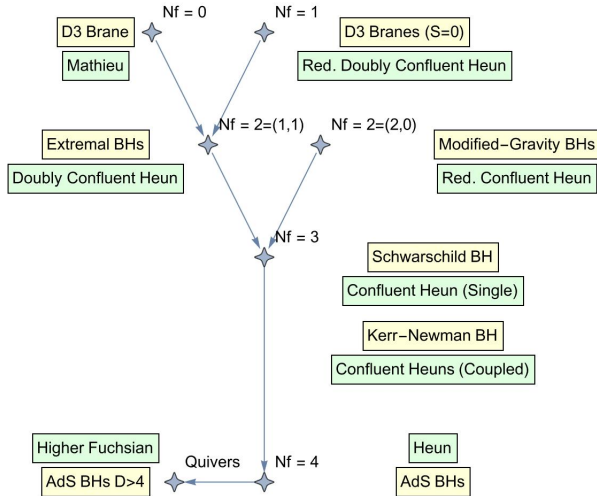


$$\begin{aligned}
 \Lambda_3 &= -16iM\omega, & u &= l(l+1) - 8M^2\omega^2 + \frac{1}{4}, & \Downarrow & & r &= \frac{4M}{\Lambda_3} e^{-y}, \\
 \hbar &= 1, & m_1 &= s - 2iM\omega, & m_2 &= -s - 2iM\omega, & m_3 &= -2iM\omega,
 \end{aligned}$$

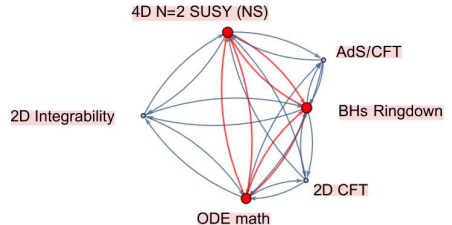
$$\left[ f(r) \frac{d}{dr} f(r) \frac{d}{dr} + \omega^2 - f(r) \left[ \frac{l(l+1)}{r^2} + (1-s^2) \frac{2M}{r^3} \right] \right] \phi(r) = 0, \tag{6}$$

$$ds^2 = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad \text{with } f(r) = 1 - 2M/r$$

# General picture of new gauge - gravity correspondence



$SU(2)$   $\mathcal{N} = 2$  (NS) gauge theories,  
Heun ODEs and BHs models.



[Aminov-Grassi-Hatsuda,  
arXiv:2006.06111,  
Bianchi-Consoli-Grillo-Morales,  
arXiv:2109.09804]

# Nekrasov-Shatashvili instanton series

- The NS prepotential can be obtained as a **series in the instanton coupling**  $\Lambda$

$$\mathcal{F}(\Lambda, a, \hbar) = \mathcal{F}^{pert}(a, \hbar) + \sum_{n=1}^{\infty} \mathcal{F}_n^{inst}(a, \hbar) \Lambda^{(4-N_f)n}. \quad (7)$$

- The **gauge period**  $a$  is recovered from the **ODE moduli parameter**  $u$  through

$$u = \Lambda \frac{\partial \mathcal{F}(\Lambda, a, \hbar)}{\partial \Lambda}. \quad (8)$$

- Also the **gauge dual instanton period**  $A_D$  is defined as

$$A_D = \frac{\partial \mathcal{F}(\Lambda, a, \hbar)}{\partial a} \quad (9)$$

and in the  $\hbar \rightarrow 0$  SW limit,  $A_D \simeq \frac{1}{\hbar} A_D^{(0)}$ , with  $A_D^{(0)} = a_D^{(0)} + 2\pi i a^{(0)}$ .

## Nice analytical expressions!

- For example for  $SU(2)$  gauge theory with  $N_f = 3$  matter flavours, at **second order** we get:

$$\begin{aligned} \mathcal{F}^{inst}(\Lambda_3, m_1, m_2, m_3, a, \hbar) = & -\frac{m_1 m_2 m_3}{4(4a^2 - \hbar^2)} \Lambda_3 \\ & + \left[ \frac{-256a^8 - 256a^6(m_3^2 + m_1^2 + m_2^2 - 2\hbar^2) + 96a^4(2m_3^2\hbar^2 + 2m_2^2(4m_3^2 + \hbar^2) + 2m_1^2(4m_3^2 + 4m_2^2 + \hbar^2) - 3\hbar^4)}{8192(a^2 - \hbar^2)(4a^2 - \hbar^2)^3} \right. \\ & + \frac{-16a^2(3m_3^2\hbar^4 + 3m_2^2(8m_3^2\hbar^2 + \hbar^4) + m_1^2(8m_2^2(10m_3^2 + 3\hbar^2) + 3(8m_3^2\hbar^2 + \hbar^4)) - 4\hbar^6)}{8192(a^2 - \hbar^2)(4a^2 - \hbar^2)^3} \\ & \left. + \frac{4m_3^2\hbar^6 + 4m_2^2\hbar^6 + 48m_3^2m_2^2\hbar^4 + 4m_1^2\hbar^2(12m_3^2\hbar^2 - 4m_2^2(28m_3^2 - 3\hbar^2) + \hbar^4) - 5\hbar^8}{8192(a^2 - \hbar^2)(4a^2 - \hbar^2)^3} \right] \Lambda_3^2 + O(\Lambda_3^3). \end{aligned} \quad (10)$$

- The computation algorithm is based on **combinatorics on Young diagrams**

Nice **analytical** expressions...but the terms are infinite and the computation time for higher orders **grows exponentially**!



# Rescued by supersymmetry?

It was found heuristically that **quasinormal modes (QNMs)** are given by quantization conditions on the gauge periods

$$A_D(\Lambda_n, \mathbf{m}, u) = 2i\pi\hbar n \quad \text{for } n \in \mathbb{N} \quad (11)$$

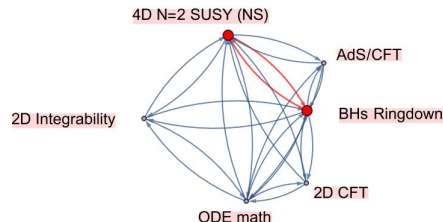
$$\Downarrow$$

$$\text{QNMs} \quad \omega_n \propto \Lambda_n$$

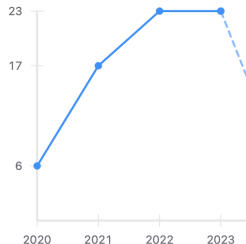
[Aminov-Grassi-Hatsuda, arXiv:2006.06111]

A small field is growing on this result, since:

- qSW theory turns out to be a **very effective method for QNMs computation**;
- it also provided **new theoretical results**.



Citations per year



# ODE/IM correspondence and BHs

# On quantum integrability

- The hallmark of quantum integrability is the presence of **infinite integrals of motion commuting** with each other

$$[\mathbf{I}_n, \mathbf{I}_m] = 0. \quad (12)$$

- The IM  $I_{2n-1}$  are also asymptotic expansion coefficients, for large spectral parameter  $\lambda = e^\theta$ , of the **Baxter's  $Q, T$  operators**

$$\ln \mathbf{Q}(\theta) \simeq -C_0 e^\theta - \sum_{n=1}^{\infty} e^{-n\theta} C_n \mathbf{I}_n \quad \theta \rightarrow +\infty, \quad (13)$$

$$[\mathbf{Q}(\theta), \mathbf{T}(\theta)] = 0, \quad (14)$$

[Bazhanov-Lukyanov-Zamolodchikov, hep-th/9412229, hep-th/9604044]

- Their eigenvalues, the  $Q, T$  **functions** satisfy **functional relations**, like **quantum wronskian  $QQ$**  or **Baxter's  $TQ$** , which given suitable **asymptotic behaviours** have **unique solutions**.

# ODE/IM correspondence essentials

- The first ODE/IM instance found was for the **anharmonic oscillator ODE**, with  $\alpha \geq -1$

$$-\frac{d^2}{dx^2}\psi(x) + \left(x^{2\alpha} + \frac{l(l+1)}{x^2} - E\right)\psi(x) = 0. \quad (15)$$

- The **spectral determinants** are also computed as **Wronskians**

$$Q^+(E, l) = Q_0^+ \prod_{k=0}^{\infty} \left(1 - \frac{E}{E_{2k}}\right) = W[\psi_{+,0}, \psi_{-,0}] \text{ with } \begin{cases} \psi_{+,0}(x) \rightarrow 0, & x \rightarrow \infty \\ \psi_{-,0}(x) \rightarrow 0, & x \rightarrow 0 \end{cases}. \quad (16)$$

- $Q^+(E, l)$  can be **computed** as **integral** of the solution  $\mathcal{P}(x) = -i \frac{d}{dx} \ln \psi(x)$  of the **Riccati** ODE equivalent to (15)

$$Q^+(E, l) = \lim_{x \rightarrow 0} [x^l \psi_{+,0}(x)] = \exp \left\{ i \int_0^{\infty} [\mathcal{P}(x) - \text{regulariz.}] dx \right\}. \quad (17)$$

- Through **discrete symmetries** of the ODE, we can define other solutions in  $\mathbb{C}$ , with  $\omega = e^{\frac{i\pi}{\alpha+1}}$

$$\Omega_+ : x \rightarrow \omega^{-1}x, \quad E \rightarrow \omega^2 E \quad \Longrightarrow \quad \psi_{+,k} \equiv \Omega_+^k \psi_{+,0} = \psi_{+,0}(\omega^{-k}x, \omega^{2k}E, l), \quad (18)$$

so we can write **lateral connection relations** in terms **Stokes multipliers**  $T_{k/2}$

$$\psi_{-1}(x) = T_{k/2}(E, l)\psi_{k-1}(x) + \tilde{T}_{k/2}(E, l)\psi_k(x) \quad \Longrightarrow \quad T_{k/2}(E, l) = W[\psi_{-1}, \psi_k]. \quad (19)$$

- Similarly defining  $\psi_{-,1}(x, E, l) \equiv \psi_{-,0}(x, E, -l-1)$  and  $Q^-(E, l) = Q^+(E, -l-1)$  from **central connection relations** (between  $\psi_{-,k}$  and  $\psi_{+,k}$ ) we obtain **QQ system**

$$\omega^{1/2}Q^+(\omega^{-1}E)Q^-(\omega E) - \omega^{-1/2}Q^+(\omega E)Q^-(\omega^{-1}E) = 2i. \quad (20)$$

- Similarly also  $TQ$  and  $T$  systems.
- $Q^\pm, T_{k/2}$  realize **vacuum  $Q, T$  functions**  
**Scaling 6-vertex IM!** [Dorey-Tateo, hep-th/9812211,  
 Bazhanov-Lukyanov-Zamolodchikov, hep-th/9812247]

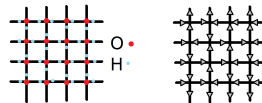
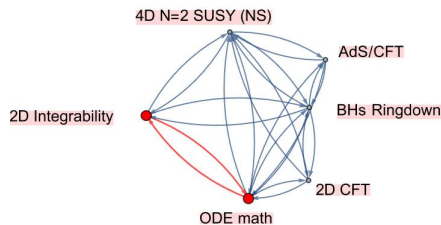


Figure: (Wikicommons/Moali)

# The ODE/IM correspondence

- The **ODE/IM correspondence** is an approach to integrability, in which the  $Q$ ,  $T$  functions and all the  $(QQ, TQ\dots)$  **integrable structures** of some Integrable Model (IM) can be **derived from an Ordinary Differential Equation** (ODE), even without knowing a priori the corresponding IM. [Dorey-Tateo, hep-th/9812211, Bazhanov-Lukyanov-Zamolodchikov, hep-th/9812247, Dorey-Dunning-Tateo, hep-th/0703066]

| ODE                    | IM               |
|------------------------|------------------|
| Spectral determinant   | $Q$              |
| Stokes multipliers     | $T_k$            |
| Central Connect. rels. | $QQ$ relation    |
| Lateral Connect. rels. | $TQ$ , $T$ -syst |
| ODE parameters         | IM parameters    |
| ODE ind. var. $x$      | ... cf. below    |



ODE/IM allows to apply integrability to very different physical theories!

# On the origin of ODE/IM

- For a long time, ODE/IM has been regarded as either **mysterious correspondence** or just a mathematical coincidence.
- However, recent work [Fioravanti-Rossi, arXiv:2106.07600] - based on [Fioravanti-Rossi-Shu, arXiv:2004.10722] - has shed light on its **origin**, by “reversing it”:

QQ/TQ/Bethe Ansatz



Marchenko-like int. eq.



Schrödinger ODEs

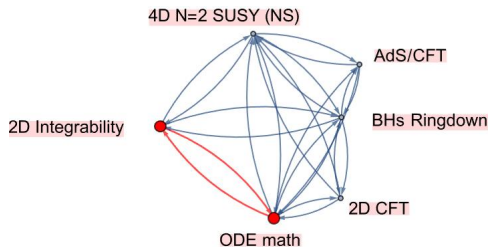


Figure: The Thinker (Rodin)

# Three different physical theories, same mathematics!

$$-\frac{d^2}{dy^2}\psi(y) + [2e^{2\theta}\cosh y + P^2]\psi(y) = 0 \quad \text{INTEGRABILITY (sd-Liouville)} \quad (21)$$

$$\frac{\hbar}{\Lambda_0} = \frac{\epsilon_1}{\Lambda_0} = e^{-\theta} \quad \frac{u}{\Lambda_0^2} = \frac{1}{2}P^2e^{-2\theta} \quad \Downarrow$$

$$-\frac{\hbar^2}{2}\frac{d^2}{dy^2}\psi(y) + [\Lambda^2\cosh y + u]\psi(y) = 0 \quad \text{N=2 GAUGE TH. (SU(2) } N_f = 0) \quad (22)$$

$$r = Le^{y/2} \quad \omega L = -2ie^\theta \quad P = \frac{1}{2}(l+2) \quad \Downarrow \quad \phi(r) = e^{y/2}\psi(y)$$

$$\frac{d^2\phi}{dr^2} + \left[ \omega^2 \left( 1 + \frac{L^4}{r^4} \right) - \frac{(l+2)^2 - \frac{1}{4}}{r^2} \right] \phi(r) = 0 \quad \text{BLACK HOLES PERT. (D3 brane)} \quad (23)$$

- We have shown this **trialeity** generally for Mathieu, Doubly Confluent Heun, Confluent Heun ODEs.



## Black holes perturbation theory

- For perturbations of the **Schwarzschild BH** metric  $g_{\mu\nu}^{(0)}$ , **Einstein equations**, after **linearization** and **gauge fixing** reduce to just a couple of much simpler PDEs:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \quad (24)$$

$$\Downarrow \quad h_{\mu\nu} = g_{\mu\nu} - g_{\mu\nu}^{(0)} \Rightarrow (Z_l, Q_l)$$

- 1) for the **axial** perturbations (phase  $(-1)^{l+1}$ ), the **Regge-Wheeler** equation

$$\left[ \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \left( 1 - \frac{2M}{r} \right) \left( \frac{l(l+1)}{r^2} - \frac{6M}{r^3} \right) \right] Q_l(t, x) = 0; \quad (25)$$

- 2) for the **polar** perturbations (phase  $(-1)^l$ ), the **Zerilli** equation

$$\left[ \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \left( 1 - \frac{2M}{r} \right) \left( \frac{72M^3}{r^5 \lambda(r)^2} + (\dots) \frac{12M}{r^3 \lambda(r)^2} \left( 1 - \frac{3M}{r} \right) + (\dots) \frac{1}{r^2 \lambda(r)} \right) \right] Z_l(t, x) = 0, \quad (26)$$

where  $\lambda(r) = l(l+1) - 2 + 6M/r$  and  $r = r(x)$ , with  $x$  tortoise coordinate.

# From PDEs to ODEs

- The **Laplace transform**  $\hat{\Psi}$  of  $\Phi = Z_I \vee Q_I$  satisfies the non-homogeneous ODE

$$\hat{\Psi}(s, x) = \int_0^\infty e^{-st} \Phi(t, x) dt, \quad \left\{ -\frac{\partial^2}{\partial x^2} + U(x) + s^2 \right\} \hat{\Psi}(s, x) = -\mathcal{I}(s, x), \quad (27)$$

where  $\mathcal{I}(s, x) = -s\Psi(t, x)|_{t=0} - \partial_t \Psi(t, x)|_{t=0}$ . The corresponding **homogeneous** equation is the **ODE we will study**:

$$\left\{ -\frac{\partial^2}{\partial x^2} + U(x) + s^2 \right\} \Psi(s, x) = 0. \quad (28)$$

- We consider **solutions** of (28) **regular** for  $x \rightarrow \pm\infty$ , that is infinity and horizon (with  $\text{Re } s > 0$ )

$$\Psi_+(s, x) \sim e^{-sx}, \quad x \rightarrow +\infty, \quad \Psi_-(s, x) \sim e^{sx}, \quad x \rightarrow -\infty. \quad (29)$$

- The Laplace tr.  $\hat{\Psi}$  is given by the **Green function**  $G$  as (with  $x_{<} = \min(x', x)$ ,  $x_{>} = \max(x', x)$ )

$$\hat{\Psi}(s, x) = \int_{-\infty}^{\infty} G(s, x, x') \mathcal{I}(s, x') dx', \quad G(s, x, x') = \frac{1}{W[\Psi_-, \Psi_+]} \Psi_-(s, x_{<}) \Psi_+(s, x_{>}). \quad (30)$$

# Mathematical definition of QNMs

- Finally taking the anti-Laplace transform of  $\hat{\Psi}$  and setting  $s = i\omega$  we get the **original perturbation**  $\Phi = Z_I \vee Q_I$  as [Nollert, Class.Quant.Grav. 16 (1999)]

$$\begin{aligned}\Phi(t, x) &= \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} e^{st} \hat{\Psi}(s, x) ds \\ &= \sum_n e^{i\omega_n t} \text{Res} \left( \frac{1}{W(s)} \right) \bigg|_{\omega_n} \int_{-\infty}^{\infty} \Psi_{-}(\omega_n, x_{<}) \Psi_{+}(\omega_n, x_{>}) \mathcal{I}(\omega_n, x') dx' + (\text{other contributions}).\end{aligned}\tag{31}$$

- The **quasinormal modes (QNMs)** are the complex frequencies (real frequencies and damping times)  $\omega_n$ , which corresponds also to **zeros of the ODE Wronskian**

$$W(\omega_n) \equiv W[\Psi_{-}, \Psi_{+}](\omega_n) = 0, \tag{32}$$

with  $W[\Psi_{-}, \Psi_{+}] = \Psi_{-}(x)\Psi'_{+}(x) - \Psi'_{-}(x)\Psi_{+}(x)$ .

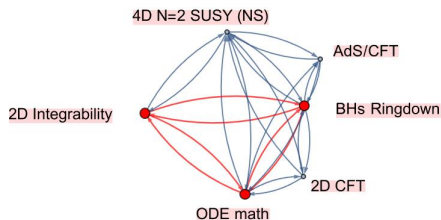
# Quasinormal modes of black holes as Bethe roots

- Through the ODE/IM formula  $Q = W[\psi_+, \psi_-]$  we proved that the **Bethe root** (zero) condition on the Baxter's  $Q$  function

$$Q(\theta_n) = 0, \quad (33)$$

is equivalent to the mathematically rigorous definition of **quasinormal modes (QNMs)**!

[Fioravanti-DG, arXiv:2112.11434]



$Q$  has **QNMs as zeroes** and satisfies a **quantum Wronskian functional relation** which can be derived for **nearly any ODE**!

- In general the QQ-system **cannot** be directly inverted and solved as **integral equation**, rather it is necessary to construct **from  $Q$  the  $Y$  function** that solves the  **$Y$  system**.

## $Y$ function and quasinormal modes

- For example, for the **triad** [ generalized Perturbed Hairpin/  $SU(2)$   $N_f = 2$  NS gauge th./ generalized extremal RN BHs ] - cf. (3) - one can define a  **$Y$  function** as  
[Fioravanti-DG, arXiv:2112.11434; Fioravanti-DG-Shu, arXiv:2208.14031]

$$Y_{+,+}(\theta) = e^{i\pi(q_1 - q_2)} Q_{+,+}(\theta) Q_{-,-}(\theta) . \quad (34)$$

- In terms of  $Y$  the  **$QQ$  system** is written as

$$e^{i\pi(q_1 + q_2)} Q_{+,+}(\theta + \frac{i\pi}{2}) Q_{-,-}(\theta - \frac{i\pi}{2}) = 1 + Y_{+,-}(\theta) . \quad (35)$$

and for **QNMs** follow the alternative **quantizations conditions**

$$Y_{+,-}(\theta_n - i\pi/2) = -1 \quad Y_{-,+}(\theta_n + i\pi/2) = -1 . \quad (36)$$

- From the  **$QQ$  system** it follows the  **$Y$  system** as

$$Y_{+,-}(\theta + \frac{i\pi}{2}) Y_{-,+}(\theta - \frac{i\pi}{2}) = [1 + Y_{+,+}(\theta)][1 + Y_{-,-}(\theta)] . \quad (37)$$

# Thermodynamic Bethe Ansatz

- Defining as the **pseudoenergy**

$$\varepsilon_{\pm,\pm}(\theta) = -\ln Y_{\pm,\pm}(\theta), \quad (38)$$

- the  $Y$  system can be inverted in the **Thermodynamic Bethe Ansatz (TBA)** for it

$$\begin{aligned} \varepsilon_{\pm,\pm}(\theta) &= \frac{8\sqrt{\pi^3}}{\Gamma(\frac{1}{4})^2} e^\theta \mp i\pi(q_1 - q_2) - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{\ln[1 + \exp(-\varepsilon_{\pm,\mp}(\theta'))] + \ln[1 + \exp(-\varepsilon_{\mp,\pm}(\theta'))]}{\cosh(\theta - \theta')} \\ \varepsilon_{\pm,\mp}(\theta) &= \frac{8\sqrt{\pi^3}}{\Gamma(\frac{1}{4})^2} e^\theta \mp i\pi(q_1 + q_2) - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{\ln[1 + \exp(-\varepsilon_{\pm,\pm}(\theta'))] + \ln[1 + \exp(-\varepsilon_{\mp,\mp}(\theta'))]}{\cosh(\theta - \theta')} \end{aligned} \quad (39)$$

- Technicality:**  $P$  enters as the  $\theta \rightarrow -\infty$  boundary condition

$\varepsilon_{+,+}(\theta, P) \sim 4P\theta + 2C(P, q_1, q_2)$ , as  $\theta \rightarrow -\infty$  (following from the perturbative expansion of ODE).

# Gravity $Y$ system and TBA

- In **gravity variables**  $\Sigma_k$  (combinations of four charges for **generalised RN extremal BHs**) the  $Y$  system reads

[Fioravanti-DG, arXiv:2112.11434; Fioravanti-DG-Shu, arXiv:2208.14031]

$$\begin{aligned} Y(\theta + \frac{i\pi}{2}, -i\Sigma_1, -\Sigma_2, i\Sigma_3, \Sigma_4) Y(\theta - \frac{i\pi}{2}, -i\Sigma_1, -\Sigma_2, i\Sigma_3, \Sigma_4) \\ = [1 + Y(\theta, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)][1 + Y(\theta, -\Sigma_1, \Sigma_2, -\Sigma_3, \Sigma_4)] \end{aligned} \quad (40)$$

- It can be inverted into the **TBA**

$$\begin{aligned} \varepsilon_{\pm}(\theta) &= [f_{0,+} \mp \frac{i\pi}{2} (\frac{\Sigma_1}{\Sigma_4^{1/4}} - \frac{\Sigma_3}{\Sigma_4^{3/4}})] e^{\theta} - \varphi * (\bar{L}_{\pm} + \bar{L}_{\mp})(\theta) \\ \bar{\varepsilon}_{\pm}(\theta) &= [\bar{f}_{0,+} \pm \frac{\pi}{2} (\frac{\Sigma_1}{\Sigma_4^{1/4}} + \frac{\Sigma_3}{\Sigma_4^{3/4}})] e^{\theta} - \varphi * (L_{\pm} + L_{\mp})(\theta) \end{aligned} \quad (41)$$

where we defined  $\varepsilon_{\pm}(\theta) = -\ln Y(\theta, \pm\Sigma_1, \Sigma_2, \pm\Sigma_3, \Sigma_4)$ ,  $\bar{\varepsilon}_{\pm}(\theta) = -\ln Y(\theta, \pm i\Sigma_1, -\Sigma_2, \mp i\Sigma_3, \Sigma_4)$ ,  $L = \ln[1 + \exp\{-\varepsilon\}]$ ,  $\varphi(\theta) = (2\pi \cosh(\theta))^{-1} \dots$

# Quasinormal modes from TBA

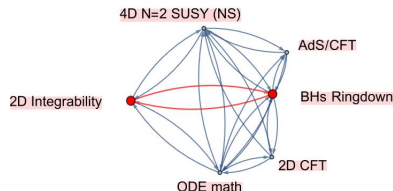
- The **QNMs condition** in gravity variables reads alternatively as

$$\begin{aligned}\bar{Y}_+(\theta_{n'} - i\pi/2) &= -1, \quad Q_{+,+}(\theta_n) = 0, \\ \bar{\varepsilon}_+(\theta_n - i\pi/2) &= -i\pi(2n+1).\end{aligned}\quad (42)$$

- Through the quantization condition on  $\bar{\varepsilon}_+$  we can **actually numerically compute QNMs from TBA**.

| $n$ | $l$ | TBA                              | Leaver                           | WKB $l \rightarrow \infty$ |
|-----|-----|----------------------------------|----------------------------------|----------------------------|
| 0   | 2   | <u>1.47799</u> - <u>0.36814i</u> | <u>1.47789</u> - <u>0.36824i</u> | 1.494 - 0.3660i            |
| 0   | 4   | <u>2.68035</u> - <u>0.36664i</u> | <u>2.68035</u> - <u>0.36664i</u> | 2.689 - 0.3660i            |
| 0   | 5   | <u>3.27959</u> - <u>0.36641i</u> | <u>3.27959</u> - <u>0.36641i</u> | 3.287 - 0.3660i            |
| 0   | 6   | <u>3.87833</u> - <u>0.36629i</u> | <u>3.87833</u> - <u>0.36629i</u> | 3.884 - 0.3660i            |
| 0   | 8   | <u>5.07501</u> - <u>0.36615i</u> | <u>5.07501</u> - <u>0.36615i</u> | 5.080 - 0.3660i            |
| 1   | 2   | <u>1.21708</u> - <u>1.12174i</u> | <u>1.22004</u> - <u>1.12585i</u> | 1.494 - 1.0979i            |
| 1   | 4   | <u>2.54243</u> - <u>1.10434i</u> | <u>2.54246</u> - <u>1.10513i</u> | 2.689 - 1.0979i            |
| 1   | 6   | <u>3.78385</u> - <u>1.10090i</u> | <u>3.78380</u> - <u>1.10122i</u> | 3.885 - 1.0979i            |
| 1   | 8   | <u>5.00305</u> - <u>1.09963i</u> | <u>5.00300</u> - <u>1.09981i</u> | 5.080 - 1.0979i            |

TABLE III. QNMs of (23) through (33) and other methods, with  $\Sigma_1 = \Sigma_3 = 0.2$ ,  $\Sigma_2 = 0.4$ ,  $\Sigma_4 = 1$ .





## Solving quantum Wronskians: D3 brane

- The scalar perturbation on the **D3 brane**, obeys **Mathieu ODE** - slide 24 - a particular case of DCHE (3), which is the same as  $SU(2)$   $N_f = 0$  quantum SW curve.
- Through linear connection relations among solutions  $\psi_k$  one easily derives the **quantum Wronskian** (or QQ-system)

$$Q(\theta + i\pi/2, a)Q(\theta - i\pi/2, a) - Q(\theta, a)^2 = 1. \quad (43)$$

- From the QQ system follows a **closed formula for Q**

$$Q(\theta, a) = \frac{\sinh \left[ \frac{1}{\hbar} A_D(\theta, a) \right]}{\sin \left[ \frac{2\pi}{\hbar} a \right]}, \quad \text{with} \quad A_D(\theta + i\pi/2, a) = A_D(\theta, a) - 2\pi i a, \quad (44)$$

We proved this and the following formulae purely from **ODE/IM and gauge theory**, but **alternative derivation** is possible through **AGT duality**

[Fioravanti-DG-Mahanta-Rossi, arXiv:24\*\*.\*\*\*\*\*]

# Solving quantum Wronskians: general results

- We proved the **exact closed general formulae** for  $Q$  (1) function for  $N_f = 0, 2, 3$  ensues

$$Q_1(\theta, m_1, m_2, a) = \exp \left\{ \frac{1}{\hbar} \sum_{k=1}^2 \frac{\partial \mathcal{F}(\theta, m_1, m_2, a)}{\partial m_k} \right\} \frac{\sinh \frac{1}{\hbar} A_D(\theta, m_1, m_2, a)}{\sin \frac{2\pi}{\hbar} a}. \quad (45)$$

(there are also other  $Q_2, Q_3$ , for  $N_f = 3$  and similarly more for  $N_f = 4$ )

- Then, it follows immediately an (at least), mathematical **mathematical proof** of the **SW-BHs correspondence**:

$$Q_1 = 0 \quad \Longleftrightarrow \quad A_D = 2i\pi\hbar n. \quad (46)$$

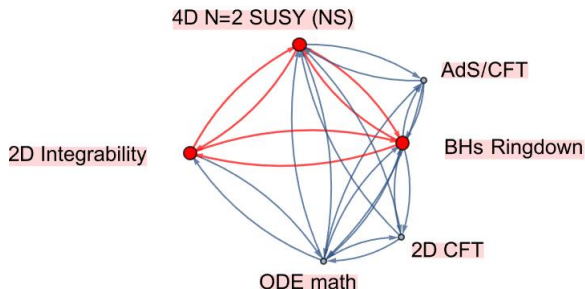
[Fioravanti-DG-Mahanta-Rossi, arXiv:24\*\*.\*\*\*\*\*]

- (46) was previously found heuristically and allowed:
  - a **novel analytic characterization of QNMs**;
  - to **find new results for the BHs theory** (by many other authors);
  - an **unexpected application of Supersymmetry**.

## Conclusions

# Summary

- We have shown **BHs perturbation theory** has a natural connection with
  - 4D  $\mathcal{N} = 2$  **NS SUSY gauge theory**;
  - 2D **integrable models**.
- In particular **ODE/IM correspondence** allowed
  - to **find new results** on all sides, at the **exact level**;
  - a **new exact method of computation of QNMs**: the Thermodynamic **Bethe Ansatz** (TBA);
  - a (mathematical) **proof** of the **SW-QNM correspondence**.



# Future developments

We notice that **much of the BH theory seems to go in parallel to the ODE/IM correspondence**, beyond the determination of QNMs!

- $\mathcal{N} = 2$  gauge theories and BHs constitute **novel fields of application of integrability** and many future developments could be pursued:
  - generalization to **more BHs models**;
  - extension to **other BHs observables**;
  - addressing **theoretical problems in BH physics**;
  - tackling **non-linear** perturbation theory;
  - **connection to other fields** (like holography).

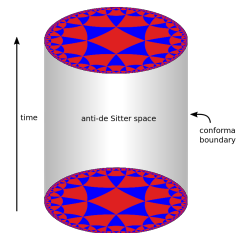
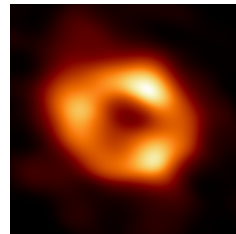
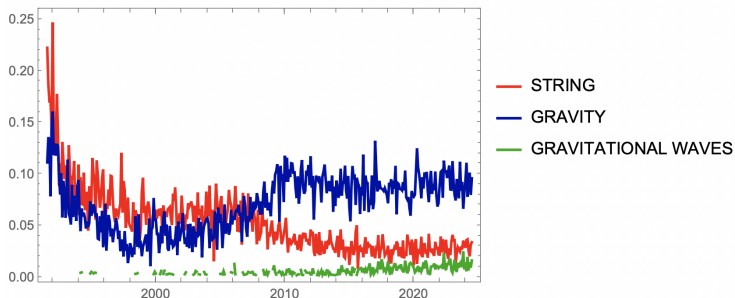


Figure: (Wikicommons/Herzog,Dunkel)

# Thank you for your attention!

```
ArXivPlot[{"string", "gravity", "gravitational waves"},
  {"hep-th", All}, "Titles", ...]
```

## Word share trends in hep-th titles



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