Quantum Integrability as a new method for exact results on black holes' observables and N=2 supersymmetric gauge theories

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and arXiv:22**.***** with also Hongfei Shu

Outline

Outline (cont.)

- AGT duality for ODE
- BPZ equation
- Connection problem
- Proofs of 4D $\mathcal{N} = 2$ SYM quantization conditions

- New results for Kerr(-dS) BHs
  - New results from $\mathcal{N} = 2$ SYM
  - New results from 2D CFT

3. Mathematical definition of quasinormal modes

4. A quantum integrable model for black holes
   - The ODE/IM procedure (1st part)
   - Quasinormal modes from TBA

5. Explanation of connection to $\mathcal{N} = 2$ gauge theory
   - Introduction to $\mathcal{N} = 2$ gauge theory
   - Integrability/Gauge correspondence (1st part)

Outline (cont.)

- Quasinormal modes from gauge periods quantization conditions
- The ODE/IM procedure (2nd part)
- Integrability/Gauge correspondence (2nd part)
- Alternative quantization condition for quasinormal modes
- Interpretation of integrability duality

6. Comparison of methods of computation of quasinormal modes

7. Generalization and perspectives

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Introduction

GW: a probe for new physics

- The recent **experimental verification of gravitational waves** renewed the **interest in the theoretical studies** of General Relativity and black hole physics. [4]
- Direct detection of gravitational waves (GW) allows to **test General Relativity** (GR) in extreme regimes and to **search for possible deviations** from it, that is, new physics. [3]
- In particular, it becomes possible to **discriminate Black Holes (BHs) from fuzzballs or other exotic compact objects (ECOs)** in terms of their multipoles, shadows or tidal effects. [3]

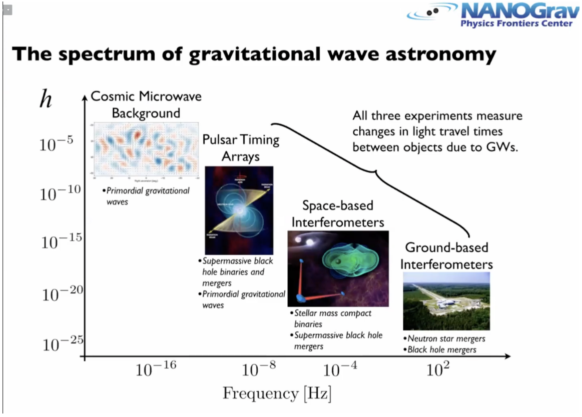


Figure: Credit: Xavier Siemens / Strings 2021

Colliding BH, ringdown phase: QNM frequencies and Spin-weighted spheroidal harmonics

- A **black hole collision** can be divided in 3 phases: **inspiral, merger and ringdown**.
- The **quasinormal modes (QNMs)** are responsible for the **damped oscillations** appearing, for example, in the **ringdown phase** of two colliding BH and have a direct connection to **gravitational waves observations**. [1]

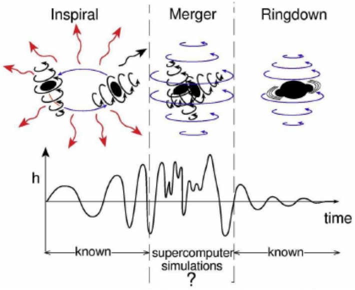


Figure: Credit: Kip Thorne

On quasinormal modes of black holes

- Quasinormal modes (QNMs) are in some sense the "characteristic sound" of black holes, that is harmonic oscillations that show up in the gravitation wave signal.
- The frequency and damping of these oscillations depend only on the parameters of the black hole.
- They essentially involve the spacetime metric outside the horizon, but also possible matter fields.

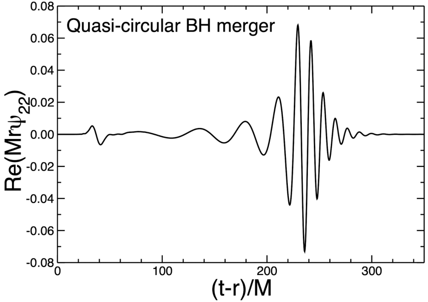
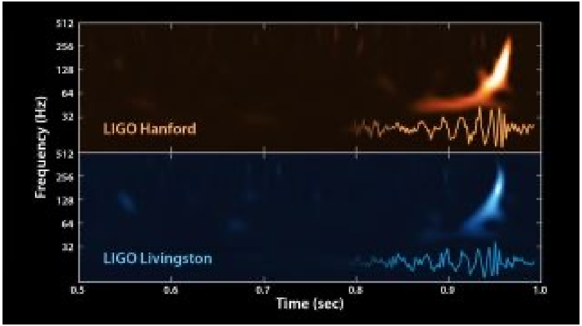


Figure: (Berti-Cardoso-Starinets:2009)

Quasinormal modes and gravitational waves

- Quasinormal modes are seen to dominate the gravitational radiation wave forms at certain times (ringdown phase), even though they are a concept born from linearized perturbation calculations.
- BH QNMs can be used to infer their mass and angular momentum and to test the no-hair theorem of general relativity.



New physics from QNM

- In the case of **Exotic Compact Objects (ECOs) in alternative theories of gravity** and **fuzzballs in string theory**, the "prompt ring-down signal" decomposes in the early-stage resonant modes, produced around their "photon-spheres" that **may differ from the BH** ones [3], because of **horizon scale structure** [7].
- At later stages both ECOs and fuzzballs produce a peculiar train of **echoes**, probing their internal "cavity" and not only their external "walls", with **significant deviations from GR**. [3]

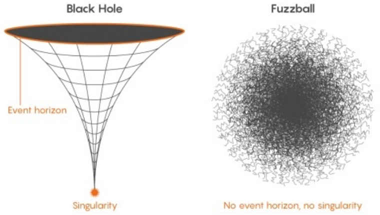


Figure: Credit: Olena Shmahalo/Quanta Magazine

- The **crucial role played by QNMs in discriminating** BHs from fuzzballs or other ECOs motivated renewed effort in their **determination with higher and higher accuracy**. [3]
- One immediate outcome of the analysis of [1, 3, 4] is a **better knowledge of the QNMs** (for the BHs considered).

Quasinormal modes and holography

- According to AdS/CFT, equilibrium and non-equilibrium properties of strongly coupled thermal gauge theories are related to the physics of higher-dimensional BHs and black branes and their fluctuations.
- Quasinormal spectra of the dual gravitational backgrounds give the location of the poles of the retarded correlators in the gauge theory, supplying important information about the theory's quasiparticle spectra and transport coefficients.
- The duality offers a new perspective on notoriously difficult problems, such as the BH information loss paradox, the nature of BH singularities and quantum gravity. Holographic approaches to these problems often involve QNMs.

From gauge to gravity and back

Classical SW curve and gauge periods

- Given an operator $H(\hat{x},\hat{p})$, with $[\hat{x},\hat{p}]=i\hbar$, we associate it with a **classical curve and a one form**
$$H(x,p)=E\,,\qquad \lambda(x,E)=p(x,E)\,d\,x \tag{1}$$
- We consider operators $H(\hat{x},\hat{p})$ such that the corresponding classical curve (120) coincides with a **SW curve** of a suitable **4D** $\mathcal{N}=2$ **gauge theory**.
- If the curve (120) has genus g, we choose a basis of cycles $\mathcal{A}_i,\mathcal{B}_i$, with $i=1,\ldots,g$ and define classical periods by integrating over such cycles
$$\Pi_{A_i,B_i}(E)=\oint_{\mathcal{A}_i,\mathcal{B}_i}p(x,E)\,d\,x\,,\qquad i=1,\ldots,g\,. \tag{2}$$
- (123) are the **SW periods** and they encode the **central charges and masses of the BPS particles** in the theory.

Quantum SW curve and gauge periods from resummation

- **Quantum Seiberg Witten curve** in the **Nekrasov-Shatashvili limit** of the Ω-background (used to compute instanton contributions) $\epsilon_2 \to 0, \epsilon_1 = \hbar$ [1] (see for more details slide **??**)

$$H_{N_f}\psi(x) = E\psi(x) \tag{3}$$

$$H_{N_f} = -\hbar^2 \frac{d^2}{dx^2} + \frac{1}{2}\Lambda_{N_f}^{2-N_f/2}\left[e^x K_+\left(\hbar\frac{d}{dx} - u_1 + \frac{\hbar}{2}\right) + e^{-x}K_-\left(\hbar\frac{d}{dx} - u_1 - \frac{\hbar}{2}\right)\right] - u_0 \ . \tag{4}$$

- Define exact periods by **resummation** of their $\hbar \to 0$ asymptotic expansions. [1]

$$\Pi^{\mathrm{WKB}}_{A_i,B_i}(E,\hbar) \doteq \sum_{n\geq 0} \hbar^n \oint_{\mathcal{A}_i,\mathcal{B}_i} Q_n(x,E)\, dx \quad \Longrightarrow \quad \Pi_{A_i,B_i} \tag{5}$$

- In practice, resummation can be done either through **TBA** or through **instanton counting**. [1] chose the latter.
- The need of resummation is **why supersymmetric gauge theory plays a crucial role** in the problem. [1]

Gauge periods from instanton calculus

- The **quantum periods** are encoded by: [1]
  - the **Nekrasov-Shatashvili free energy** $\mathcal{F}^{(N_f)}(a, \mathbf{m}, \Lambda_{N_f}, \hbar)$, which can be computed systematically (at least at the instanton level) through combinatorial calculus on Young Tableaus of the gauge group representation;
  - the **4D quantum mirror map** $a(E, \mathbf{m}, \Lambda_{N_f}, \hbar)$.
- That is

$$\Pi_A^{(N_f)}(E, \mathbf{m}, \Lambda_{N_f}, \hbar) = a(E, \mathbf{m}, \Lambda_{N_f}, \hbar) \tag{6}$$

and

$$\Pi_B^{(N_f)}(E, \mathbf{m}, \Lambda_{N_f}, \hbar) = \partial_a \mathcal{F}^{(N_f)}(E, \mathbf{m}, \Lambda_{N_f}, \hbar)\Big|_{a=a(E,\mathbf{m},\Lambda_{N_f},\hbar)} . \tag{7}$$

- The **Matone's relation**

$$E = a^2 - \frac{\Lambda_{N_f}}{4 - N_f} \frac{\partial \mathcal{F}_{\text{inst}}^{(N_f)}(a; \mathbf{m}, \Lambda_{N_f}, \hbar)}{\partial \Lambda_{N_f}} . \tag{8}$$

can be inverted leading to the four dimensional quantum mirror map [1].

Quantization condition, resonances

- The discrete spectrum of H_{N_f} is captured by the following **quantization condition**: [1]

$$\Pi_I^{(N_f)}(E,\mathbf{m},\Lambda_{N_f},\hbar)=\mathcal{N}_I\left(n+\frac{1}{2}\right), \qquad I=A,B, \qquad n=0,1,2,..., \tag{9}$$

- On one hand one usually imposes suitable positivity conditions on

$$\hbar\,,\Lambda_{N_f}\,,m_i \tag{10}$$

  so that H_{N_f} has a real, discrete spectrum.

- On the other hand if $\hbar\,,\Lambda_{N_f}\,,m_i$ are **complex**, we can think of the spectral problem in terms of **resonances**. Since BH QNMs are nothing but resonances, **we strongly expect** that their spectra are computed in the geometric/gauge framework. [1]

General recipe

The **recipe** for setting up the correspondence is the following: [1]

1. read off the **singularity structure** of the ODE;
2. look for **gauge theory counterparts** by comparing the Riemann sphere with punctures associated with quiver gauge theories;
3. find relations between the **parameters**.

2D CFT: physical explanation?

- In the view of [4], the **2D CFT** framework, connected to 4D $\mathcal{N}=2$ SYM through **AGT duality**, is the suitable one to provide a **physical explanation of the relation between black hole physics and supersymmetric gauge theories**.
- By using the explicit solution of the connection problem, [4] provide a **proof of the exact quantization of Kerr black hole quasinormal modes** as proposed in [1]. By solving the angular Teukolsky equation, [4] also **prove the analogue dual quantization condition on the corresponding parameters of the spin-weighted spheroidal harmonics** (see section 3 for further explanations).
- [6] argue that the use of the AGT correspondence may also provide **additional information in discriminating BHs from fuzzballs**.

Schwarzschild BH / $SU(2)$ $N_f = 3$ $\mathcal{N} = 2$ SYM - I

- Schwarzschild metric for **static and spherically symmetric solutions** to the Einstein equation in the vacuum

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{11}$$

- Scalar ($s = 0$), electromagnetic ($s = 1$) and odd-parity gravitational ($s = 2$) **linear perturbations of the metric** are governed by the **Regge-Wheeler type equation**

$$\left[f(r)\frac{d}{dr}f(r)\frac{d}{dr} + \omega^2 - V(r)\right]\phi(r) = 0\,, \tag{12}$$

$$f(r) = \left(1 - \frac{2M}{r}\right)\,, \qquad V(r) = f(r)\left[\frac{l(l+1)}{r^2} + (1 - s^2)\frac{2M}{r^3}\right]\,, \tag{13}$$

$$\text{boundary conditions} \qquad \phi(r) \sim \begin{cases} e^{-i\omega(r+2M\ln(r-2M))} & r \to 2M \\ e^{+i\omega(r+2M\ln(r-2M))} & r \to \infty \end{cases}\,. \tag{14}$$

Schwarzschild BH / $SU(2)$ $N_f = 3$ $\mathcal{N} = 2$ SYM - II

- By the change of variable [1]

$$r = 2Mz \,, \qquad \phi(r) = \sqrt{\frac{z}{z-1}}\Phi(z) \,, \tag{15}$$

(12) becomes the equation

$$\Phi''(z) + \tilde{Q}(z)\Phi(z) = 0 \qquad \tilde{Q}(z) = \frac{1}{z^2(z-1)^2}\sum_{i=0}^{4}\tilde{A}_i z^i \tag{16}$$

$$\tilde{A}_0 = -s^2 + \frac{1}{4} \,, \quad \tilde{A}_1 = l(l+1) + s^2 \,, \quad \tilde{A}_2 = -l(l+1) \,, \quad \tilde{A}_3 = 0 \,, \quad \tilde{A}_4 = (2M\omega)^2 \,. \tag{17}$$

- Equation (16) has two regular singularities at $z = 0, 1$ and one irregular singularity (of Poincaré rank 1) at $z = \infty$. It corresponds to the **Confluent Heun** equation.

Schwarschild BH / $SU(2)$ $N_f = 3$ $\mathcal{N} = 2$ SYM - III

- which stands in correspondence with the $SU(2)$ **quantum Seiberg-Witten curve for** $N_f = 3$ by the parameters correspondence [1]

$$\hbar = 1\,, \qquad \Lambda_3 = -16 i M\omega\,, \qquad E = -l(l+1) + 8M^2\omega^2 - \frac{1}{4}\,, \qquad m_1 = s - 2iM\omega\,, \qquad m_2 = -s - 2iM\omega\,, m_3 = -2iM\omega\,. \tag{18}$$

-

$$\left[-\hbar^2\partial_x^2 + Q_3(x)\right]\tilde{\psi}(x) = 0 \tag{19}$$

$$Q_3(x) = \frac{e^{-2x}}{16(\sqrt{\Lambda_3}e^x - 2)^2}\Big[4\Lambda_3 + 4\Lambda_3 e^{4x}(m_1 - m_2)^2 + \Lambda_3 e^{2x}(\Lambda_3 - 24 m_3) \tag{20}$$

$$+ 4\sqrt{\Lambda_3}e^{3x}(8m_1 m_2 + \Lambda_3 m_3 - 2\hbar^2) - 4\sqrt{\Lambda_3}e^x(\Lambda_3 - 8m_3)\Big] + \frac{2E}{\sqrt{\Lambda_3}e^x - 2} \tag{21}$$

Schwarzschild BH / $SU(2)$ $N_f = 3$ $\mathcal{N} = 2$ SYM - IV

$$\psi(x) = \exp\left[\frac{e^{-x}\{e^x[\Lambda_3 x - 4(m_1 + m_2 + \hbar)\ln(2 - \sqrt{\Lambda_3}e^x)] - 2\sqrt{\Lambda_3}\}}{8\hbar}\right]\tilde{\psi}(x) \tag{22}$$

$$z = \frac{2e^{-x}}{\sqrt{\Lambda_3}} \quad \tilde{\psi}(x) = z^{-1/2}\Psi(z) \tag{23}$$

$$\hbar^2\Psi''(z) + \frac{1}{z^2(z-1)^2}\sum_{i=0}^{4}\hat{A}_i z^i \Psi(z) = 0 \tag{24}$$

Schwarzschild BH / $SU(2)$ $N_f = 3$ $\mathcal{N} = 2$ SYM - V

- The **quantization condition** for $SU(2)$ $N_f = 3$ $\mathcal{N} = 2$ SYM in terms of Schwarschild BHs parameters reads [1]

$$\Pi_B^{(3)}\left(-l(l+1) + 8M^2\omega^2 - \frac{1}{4}, \mathbf{m}, -16iM\omega, 1\right) = 2\pi\left(n + \frac{1}{2}\right), \qquad n = 0, 1, 2, \dots, \tag{25}$$

$$\mathbf{m} = \{s - 2iM\omega, -s - 2iM\omega, -2iM\omega\}\ . \tag{26}$$

- For a given set of quantum numbers $\{l, s, n\}$, this equation admits a **discrete family of complex solutions** $\omega_n(l, s)$ which give the **QNMs** $M\omega_n$. [1]
-
  1. The LHS of (25) is computed by Nekrasov's combinatorial formula systematically.
  2. The problem is that the NS free energy includes the parameter a that is not directly related to the black hole parameters. To avoid it we use Matone relation (8).
  3. Therefore we can finally eliminate a from the NS free energy, and thus we can solve the quantization condition (25) with respect to $M\omega$.

Kerr BH / $SU(2)$ $N_f = 3$ $\mathcal{N} = 2$ SYM - I

- Kerr metric for **stationary and axially symmetric solutions** to the Einstein equation in the vacuum (in Boyer-Lindquist coordinates, with mass M, angular momentum α)

$$ds^2 = -dt^2 + dr^2 + 2\alpha \sin^2\theta dr d\phi + (r^2 + \alpha^2\cos^2\theta) d\theta^2 + (r^2+\alpha^2)\sin^2\theta d\phi^2 \tag{27}$$

$$+ \frac{2Mr}{r^2 + \alpha^2\cos^2\theta}(dt + dr + \alpha\sin^2\theta d\phi)^2 \tag{28}$$

- **Perturbations** of rotating black holes are described by the **Teukolsky equation**, which is a **separable PDE** in the angular and radial part [1], even though the background is neither spherically symmetric nor static. [2]
- Both the angular and radial Teukolsky equation have the same singularity structure as the **confluent Heun equation**. [1]

Kerr BH / $SU(2)$ $N_f = 3$ $\mathcal{N} = 2$ SYM - II: angular part Teukolsky

- **Angular part Teukolsky** equation:

$$\left[\frac{d}{dx}(1-x^2)\frac{d}{dx} + (cx)^2 - 2csx + {}_sA_{lm} + s - \frac{(m+sx)^2}{1-x^2}\right] {}_sS_{lm}(x) = 0 \qquad (29)$$

  $x = \cos\theta$, s is the (minus of) spin, $l = 0, 1, 2, ...$, $|m| \leq l$, $m \in \mathbb{Z}$ for integer spins, $m \in \frac{1}{2} + \mathbb{Z}$ for half integer spins, $c = \alpha\omega$.

- ${}_sS_{lm}(x)$ is the **spin-weighted spheroidal harmonics**, with **eigenvalue** ${}_sA_{lm}$ determined by the regularity condition of ${}_sS_{lm}(x)$ at $x = \pm 1$. For general s, l, m, c, no closed form of ${}_sA_{lm}$ is known so far. ${}_sA_{lm}(c = 0) = {}_sA^{(0)}_{lm} = l(l+1) - s(s+1)$.

- By changing $z = (1+x)/2$, $y(z) = \sqrt{1-x^2}\,{}_sS_{lm}(x)/2$ and

$$\Lambda_3 = 16c\,, \quad E = -{}_sA_{lm} - s(s+1) + c^2 - \frac{1}{4} \qquad (30)$$

$$m_1 = -m\,, \quad m_2 = m_3 = -s\,. \qquad (31)$$

  we still connect to the $SU(2)$, $N_f = 3$ **quantum SW curve** equation. [1]

Kerr BH / $SU(2)$ $N_f = 3$ $\mathcal{N} = 2$ SYM - III: quant. cond. ang. Teukolsky

- The **quantization condition** this time is **also** for the A-**period** and reads [1]

$$\Pi_A^{(3)}\left(-{}_sA_{lm} - s(s+1) - c^2 - \frac{1}{4}, \mathbf{m}, 16c, 1\right) = i\left(l + \frac{1}{2}\right) \tag{32}$$

- The **difference between the A-period condition and the B-period condition** is the following. In the angular problem we impose the boundary conditions at the two regular singular points $z = 0, 1$. In the radial problem, we have to impose the conditions at the regular singular point $z = 1$ and at the irregular singular point $z = 1$.
- Equation (32) is **equivalent** to [1]

$${}_sA_{lm} - {}_sA_{lm}^{(0)} + c^2 = \Lambda_3 \frac{\partial \mathcal{F}_{\text{inst}}^{(3)}}{\partial \Lambda_3}(il + i/2, \mathbf{m}, \Lambda_3, 1)\Big|_{\Lambda_3 = 16c} \qquad \mathbf{m} = \{-m, -s, -s\} \tag{33}$$

from which we can **obtain** ${}_sA_{lm}$ **as an expansion in powers of** c.

Kerr BH / $SU(2)$ $N_f = 3$ $\mathcal{N} = 2$ SYM - IV: radial part Teukolsky

- **Radial part Teukolsky** equation:

$$\Delta(r)R''(r) + (s+1)\Delta'(r)R'(r) + V_T(r)R(r) = 0 \qquad \Delta(r) = r^2 - 2Mr + \alpha^2 \tag{34}$$

$$V_T(r) = \frac{K(r)^2 - 2is(r-M)K(r)}{\Delta(r)} - {}_sA_{lm} + 4is\omega r + 2\alpha m\omega - \alpha^2\omega^2 \quad K(r) = (r^2+\alpha^2)\omega - \alpha m$$

- Equation (34) has regular singular points at $r = r_\pm = M \pm \sqrt{M^2 - \alpha^2}$, corresponding to the event and Cauchy horizons. Boundary conditions:

$$R(r) \sim \begin{cases} (r_+ - r_-)^{-1-s+i\omega+i\sigma_+} e^{i\omega r_+}(r-r_+)^{-s-i\sigma_+} & r \to r_+ \\ A(\omega) r^{-1-2s+i\omega} e^{i\omega r} & r \to \infty \end{cases} \qquad \sigma_+ = \frac{\omega r_+ - \frac{\alpha m}{2M}}{\sqrt{1 - \frac{\alpha^2}{M^2}}}$$

- **We get** $SU(2)$ $N_f = 3$, by changing $z = (r - r_-)/(r_+ - r_-)$, $y(z) = \Delta(r)^{(s+1)/2}R(r)$, [1]

$$\Lambda_3 = -16i\omega\sqrt{M^2 - \alpha^2} \quad E = -{}_sA_{lm} - s(s+1) + (8M^2 - \alpha^2)\omega^2 - \frac{1}{4} \tag{35}$$

$$m_1 = s - 2iM\omega\,, \quad m_3 = -s - 2iM\omega\,, \quad m_2 = i(-2M^2\omega - \alpha m)/\sqrt{M^2 - \alpha^2}\,. \tag{36}$$

Kerr BH / $SU(2)$ $N_f = 3$ $\mathcal{N} = 2$ SYM - III: quant. cond. rad. Teukolsky

- The radial equation has the **discrete set of complex frequencies** $\omega_n(l, s, m)$ - **QNMs** $M\omega_n$ - given by the **quantization condition for the** B**-period**

$$\Pi_B^{(3)}\left(-{}_sA_{lm} - s(s+1) + (8M^2 - \alpha^2)\omega^2 - \frac{1}{4}, \mathbf{m}, -16i\omega\sqrt{M^2-\alpha^2}, 1\right) = 2\pi\left(n + \frac{1}{2}\right), \tag{37}$$

$$\mathbf{m} = \left\{s - 2iM\omega, \frac{i(-2M^2\omega - \alpha m)}{\sqrt{M^2-\alpha^2}}, -s - 2iM\omega\right\}, \tag{38}$$

where ${}_sA_{lm}$ are given by (33).

- Thus through the combination of the angular and radial quantization condition it is possible to **find both** ${}_sA_{lm}$ **and** ω_n **by using quantities in** $\mathcal{N} = 2$ **SW theory**. [1]

Extremal limit Kerr BH / $SU(2)$ $N_f = 2$ $\mathcal{N} = 2$ SYM - I

- The **extremal limit in Kerr BHs** corresponds to $\alpha \to M$, which means in gauge theory

$$\Lambda_3 \to 0\,, \quad m_2 \to \infty\,, \quad \Lambda_3 m_2 = -16M\omega(2M\omega - m) = \text{fixed}\,, \tag{39}$$

  that is, the **decoupling limit under which the $N_f = 3$ theory flows to the $N_f = 2$ one**. [1]

- The radial part of the Teukolsky equation corresponds then to the **double confluent Heun** equation, quantum SW curve of $SU(2)$ $N_f = 2$ **SYM** under the dictionary: [1]

$$\Lambda_2^2 = -16M\omega(2M\omega - m) \qquad E = -{}_sA_{lm} - s(s+1) + 7M^2\omega^2 - \frac{1}{4} \tag{40}$$

$$m_1 = s - 2iM\omega \qquad m_2 = -s - 2iM\omega\,. \tag{41}$$

Extremal limit Kerr BH / $SU(2)$ $N_f = 2$ $\mathcal{N} = 2$ SYM - II

- In the extremal limit, the **radial quantization condition** for the B-period for the $N_f = 2$ theory **gets simplified** [1]

$$\Pi_B^{(2)}\left(-{}_sA_{lm} - s(s+1) + 7M^2\omega^2 - \frac{1}{4}, \mathbf{m}, -4i\sqrt{\omega(2M^2\omega - Mm)}, 1\right) = 2\pi\left(n + \frac{1}{2}\right), \tag{42}$$

$$\mathbf{m} = \{s - 2iM\omega, -s - 2iM\omega\} \tag{43}$$

(A)dS Schwarschild BH / $SU(2)$ $N_f = 4$ & $SU(2) \times SU(2)$ quiver $\mathcal{N} = 2$ SYM

- Consider **4D asymptotically (A)dS Schwarschild BHs**. The radial master equation has the same form as (12) with

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 , \quad V(r) = f(r) \left[\frac{l(l+1)}{r^2} + (1 - s^2) \left(\frac{2M}{r^3} - \frac{4 - s^2}{6} \Lambda \right) \right] \tag{44}$$

where Λ is the cosmological constant. So after the redefinition of $\phi(r) = \Phi(r)/\sqrt{f(r)}$ [1]

$$\Phi''(r) + \left[\frac{\omega^2 - V(r)}{f(r)^2} + \frac{f'(r)^2}{4f(r)^2} - \frac{f''(r)}{2f(r)} \right] \Phi(r) = 0 \tag{45}$$

- The 3 roots of $f(r) = 0$ as well as $r = 0$ are regular singular points. For $s = 1, 2$, $r = \infty$ is not a singular point, but for $s = 0$ it is a regular singular point.
- **For** $s = 1, 2$ the master eq. is a **Heun equation** and it corresponds to **gauge theory with** $N_f = 4$. The singularity structure **for** $s = 0$ is realized in an $SU(2) \times SU(2)$ **quiver gauge theory**. [1]

Higher dimensional (A)dS / $SU(2)^{D-2}$ & $SU(2)^{D-3}$ quiver gauge theories

- [1] consider also **higher D-dimensional asymptotically (A)dS BHs**.
  - **For $D > 4$ and $\Lambda \neq 0$** the quantum SW curve having the same singularity structure as the master equation (**Fuchsian ODE with $D+1$ singular points**) is an $SU(2)^{D-2}$ **quiver gauge theory**.
  - **For $\Lambda = 0$** there is an **irregular singular point** with Poincaré rank one, so we have to consider $SU(2)^{D-3}$ **quiver gauge theories** associated with the **Riemann sphere with an irregular puncture**.
- We notice that the **dimensional information of the black hole** is reflected in the **number of quiver gauge groups**. [1]

D3 brane BH / $SU(2)$ $N_f = 0$ $\mathcal{N} = 2$ SYM - I

- **D3 brane** metric

$$ds^2 = H(r)^{-1/2}(-dt^2 + d\mathbf{x}^2) + H(r)^{1/2}(dr^2 + r^2 d\Omega_5^2) \tag{46}$$

where $\mathbf{x}$ are the longitudinal coordinates, $H(r) = 1 + \frac{L^4}{r^4}$, $d\Omega_5^2$ denotes the metric of the transverse round S_5 sphere.

- The master equation in the variable $z = -i \ln \frac{r}{L}$ is the **Mathieu equation** [3]

$$\Phi''(z) + [\alpha - 2\beta \cos(2z)]\,\Phi(z) = 0\,. \tag{47}$$

- The parameter change to the $SU(2)$ $N_f = 0$ **gauge theory** is [3]

$$\alpha = \frac{4u}{\hbar^2} = (l+2)^2\,, \quad \beta = \frac{4\Lambda^2}{\hbar^2} = \omega^2 L^2\,. \tag{48}$$

D3 brane BH / $SU(2)$ $N_f = 0$ $\mathcal{N} = 2$ SYM - II

- The A-**period** is given by the **Floquet exponent**

$$a(\hbar, u, \Lambda) = \frac{\hbar}{2}\nu(\alpha, \beta)\,. \tag{49}$$

- The QNMs $\omega(l, n)$ can be obtained by solving numerically the **quantization condition** [3]

$$\nu = \frac{n}{2}\,. \tag{50}$$

- **Particular limits** of the Floquet exponent can be obtained from the expansions for α

$$\alpha = \nu^2 + \frac{\beta^2}{2(\nu^2-1)} + \frac{(5\nu^2+7)\beta^4}{32(\nu^2-1)^3(\nu^2-4)} + \dots \qquad \beta \ll \alpha \quad \text{(weak coup.)} \tag{51}$$

$$\alpha = -2\beta + 4\nu\sqrt{\beta} - \frac{1}{8}(1+4\nu^2) - \frac{\nu(4\nu^2+3)}{2^6\sqrt{\beta}} + \dots \quad \alpha \simeq \pm 2\beta \quad (u \simeq \pm 2\Lambda^2) \quad \text{(photon sp.)} \tag{52}$$

AGT duality for ODE

- According to **AGT duality**, **conformal blocks** of Virasoro algebra for **2D CFT** can be identified with concrete combinatorial formulae arising from equivariant **instanton counting** in the context of $\mathcal{N} = 2$ **4D supersymmetric gauge theories**.
- 2D CFT and of the representations of its Virasoro symmetry algebra provide **exact solutions to PDEs** in terms of conformal blocks and appropriate fusion coefficients. [4]
  - The prototypical example is the **null-state equation at level** 2 for primary operators of Virasoro algebra which reduce, in the large central charge limit, to a **Schrödinger-like equation** with regular singularities, corresponding to a potential term with at most quadratic poles. [4]
  - The wave function $\Psi(z)$ corresponds to the insertion of a BPS surface observable in the gauge theory path integral. The relevant gauge theory moduli space in these cases is the one of ramified instantons, with vortices localised on the surface defect, the z-variable providing the fugacity for the vortex counting.
  - The advantage of this approach is that the **explicit solution of the connection problem** on the z-plane for the equation can be derived from the explicit computation of the **full** CFT_2 **correlator** and from its **expansions in different intermediate channels**. [4]

Confluent Heun equation as BPZ equation

- The representation theory of the Virasoro algebra for central charge $c = 1 + 6Q^2$, $Q = b + \frac{1}{b}$, contains degenerate Verma modules of weight $\Delta_{r,s} = \frac{Q^2}{4} - \alpha_{r,s}^2$ with $\alpha_{r,s} = -\frac{br}{2} - \frac{s}{2b}$.
- At level 2, the degenerate field $\Phi_{2,1}$ satisfies the **null state equation**

$$(b^{-2}L_{-1}^2 + L_{-2})\Phi_{2,1}(z) = 0 \tag{53}$$

  It translates into a differential equation, the **BPZ equation for correlation functions**.
- Consider the **chiral correlator with a degenerate field insertion**

$$\Psi(z) = \langle \Delta, \Lambda_0, m_0 | \Phi_{2,1}(z) V_2(1) | \Delta_1 \rangle \tag{54}$$

  where $|\Delta_1\rangle$ is a primary state and $\langle \Delta, \Lambda_0, m_0|$ is an irregular state of rank 1 [4].

Confluent Heun equation as BPZ equation (cont.)

- On the resulting BPZ equation, we take the **Nekrasov-Shatashvili (NS) limit** in the AGT dual gauge theory, corresponding to the **semiclassical limit of large Virasoro central charge $c \to \infty$ in the CFT**. To this end, we introduce a new parameter $\hbar$, send $\epsilon_2 = \hbar b \to 0$ and keeping $\epsilon_1 = \hbar / b$, $\hat{\Delta} = \hbar^2 \Delta$, $\hat{\Delta}_k = \hbar^2 \Delta_k$, $\Lambda = 2i\hbar\Lambda_0$, $m_3 = \frac{i}{2}\hbar m_0$ fixed. Introducing the normalized wave function $\psi(z) = \lim_{\epsilon_2 \to 0} \Psi(z) / \langle \Delta, \Lambda_0, m_0 | V_2(1) | \Delta_1 \rangle$ the BPZ equation becomes

$$\left[\epsilon_1^2 \partial_z^2 - \frac{1}{z}\frac{1}{z-1}(-\Lambda\partial_\Lambda \mathcal{F}^{inst} + \hat{\Delta}_2 + \hat{\Delta}_1 - \hat{\Delta}) + \frac{\hat{\Delta}_2}{(z-1)^2} + \frac{\hat{\Delta}_1}{z^2} - \frac{m_3\Lambda}{z} - \frac{\Lambda^2}{4}\right]\psi(z) = 0 \tag{55}$$

Confluent Heun equation as BPZ equation (cont.)

- It takes the form of a **Schrödinger equation**

$$\epsilon_1^2 \frac{d^2\psi(z)}{dz^2} + \frac{1}{z^2(z-1)^2}\sum_{i=0}^{4} A_i z^i = 0 \tag{56}$$

Using conformal momenta instead of dimensions we write $\hat{\Delta}_i = \frac{1}{4} - a_i^2$. with $a_i = \hbar\alpha_i$. Defining furthermore $E = a^2 - \Lambda\partial_\Lambda \mathcal{F}^{inst}$ the coefficients of the potential are

$$A_0 = \frac{\epsilon_1^2}{4} - a_1^2 \qquad A_1 = -\frac{\epsilon_1^2}{4} + E + a_1^2 - a_2^2 - m_3\Lambda \tag{57}$$

$$A_2 = \frac{\epsilon_1^2}{4} - E + 2m_3\Lambda - \frac{\Lambda^2}{4} \qquad A_3 = -m_3\Lambda + \frac{\Lambda^2}{2} \qquad A_4 = -\frac{\Lambda^2}{4} \tag{58}$$

Confluent Heun equation as BPZ equation (cont.)

- The **dictionary** for connecting to the **radial Teukolsky** for **Kerr BHs** equation is

$$E = \tag{59}$$

$$a_1 = -i\frac{\omega - m\Omega}{4\pi T_H} + 2iM\omega + \frac{s}{2} \qquad a_2 = -i\frac{\omega - m\Omega}{4\pi T_H} - \frac{s}{2} \tag{60}$$

$$m_3 = -2iM\omega + s \qquad \Lambda = -2i\omega(r_+ - r_-) \tag{61}$$

where T_H is the Hawking temperature and Ω is the angular velocity at the horizon given by

$$T_H = \frac{r_+ - r_-}{8\pi M r_+}\,, \qquad \Omega = \frac{a}{2Mr_+}\,. \tag{62}$$

Confluent Heun equation as BPZ equation (cont.)

The remaining masses are obtained by AGT

$$m_1 = a_1 + a_2 = -i\frac{\omega - m\Omega}{2\pi T_H} + 2iM\omega \tag{63}$$

$$m_2 = a_2 - a_1 = -2iM\omega - s \tag{64}$$

$$m_3 = -2iM\omega + s\,. \tag{65}$$

- On the other hand, the dictionary which connects to **angular Teukolsky** equation for **Kerr BHs** is

$$E = \frac{1}{4} + c^2 + s(s+1) - 2cs + \lambda \tag{66}$$

$$a_1 = -\frac{m-s}{2} \qquad a_2 = -\frac{m+s}{2} \tag{67}$$

$$m_3 = -s \qquad \Lambda = 4c \tag{68}$$

Confluent Heun equation as BPZ equation (cont.)

and all the masses of gauge theory are, by AGT

$$m_1 = a_1 + a_2 = -m\,, \quad m_2 = a_2 - a_1 = -s\,, \quad m_3 = -s\,. \tag{69}$$

The connection problem

- Exploiting **crossing symmetry of Liouville correlation functions** we can **connect different asymptotic expansions** of the solutions of BPZ equations around different field insertion points. [4]
- The irregular correlators appearing in such asymptotics can be efficiently computed as **Nekrasov partition functions** thanks to the AGT correspondence (in the NS limit $\epsilon_2 \to 0, \epsilon_1 = 1$).
- In particular, the connection matrix M is defined as the matrix which relates solution of the confluent Heun equation around 1 and ∞ (t and u channel) which give the irregular correlator
$$f_i^{(t)}(z) = M_{ij} f_j^{(u)}(z) \qquad i = \alpha_{2\pm}\,, \quad j = \alpha_{\pm}\,. \tag{70}$$
- There are several interesting **physical quantities in the black hole** problem which are governed by the Teukolsky equation. Having the **explicit expression for the connection coefficients** allows [4] to **compute them exactly**. See next subsection and section 4.

Proof of quantization conditions

- From the explicit expression of the connection matrix M, [4] extract the quantization condition for the quasinormal modes, by **imposing that the coefficient of the ingoing wave vanishes**.

$$\partial_a \mathcal{F} = 2\pi i n \,, \qquad n \in \mathbb{Z} \,, \tag{71}$$

which corresponds to (37).

- [4] compute also of the angular eigenvalue. To that end, they impose that the **angular eigenfunctions be regular at** $z = 0, 1$, relative to another connection matrix N. The divergences are cured if

$$a = -i\Pi_A^{(3)} = l + \frac{1}{2} \,, \qquad l \geq \max\left(m, s\right) , \tag{72}$$

which corresponds to (32).

New result from $\mathcal{N}=2$ SYM: an alternative to Kerr Teukolsky equation

- Consider **Kerr BHs**. The **radial Teukolsky equation** is complicated and uncoventional. This fact often prevents us from understanding its analytic properties. [2]
- [2] found a **much simpler** description through a **new alternative equation**

$$\left[f(z)\frac{d}{dz}f(z)\frac{d}{dz}+(2M\omega)^2-V(z)\right]\phi(z)=0 \quad \text{with} \quad f(z)=1-\frac{1}{z} \tag{73}$$

$$V(z)=f(z)\left[4c^2+\frac{4c(m-c)}{z}+\frac{{}_sA_{lm}+s(s+1)-c(2m-c)}{z^2}-\frac{s^2-1}{z^3}\right] \tag{74}$$

- The alternative equation (73) is **isospectral** to the original one. Besides, it has the same form as the master equations for spherically symmetric BHs if c and ${}_sA_{lm}$ are regarded as given parameters [2].
- Once moving to the $\mathcal{N}=2$ SYM, we have a **symmetry for flavor masses** respected by the **QNM spectrum**. However, the same symmetry is **not manifest** at the level of differential equations. [2]
- The **physical origin** of equation (73) in BH perturbation theory is **not known**. [2]

General modal stability problem for Kerr(-dS) BHs

- As **Kerr(-de Sitter)** black holes are stationary spacetimes, they correspond to equilibrium states for Einsten equations, and one would like to **determine** whether they are **stable or unstable equilibria**. [5]
- Outside very special classes, stability of Kerr(-de Sitter) is an **open question**. [5]
- If Kerr-(dS) BHs are nonlinearly stable, the most basic statement one can hope to prove for Teukolsky equation is that it is **modally stable**, i.e. that there are no separable solutions to it which are exponentially growing or bounded and non-decaying in time. By separable solution we mean a solution of the form

$$\alpha^{[s]}(t,r,\theta,\phi) = e^{-i\omega t} e^{im\phi} S(\theta) R(r)\,, \tag{75}$$

  where S and R satisfy, respectively, the angular and radial Teukolsky ODE. [5]
- For Kerr-(dS) BHs there are **many possible sources of mode instabilities** (such as *superradiance*). [5]
  - It is remarkable that, in the $\Lambda = 0$ **case, mode stability holds for the entire Kerr black hole family** with angular momentum $|a| \leq M$.
  - In the $\Lambda > 0$ **Kerr-dS** case modal stability remains an **open problem** for general parameters.

New result from $\mathcal{N} = 2$ SYM: proofs of mode stability for Kerr(-dS) BHs

- **Thanks to the correspondence with $\mathcal{N} = 2$ theories** discovered first by [1], [5] gave
  1. a new alternative proof of the classic mode stability result for Kerr BHs
  2. a **completely new (though partial in the parameter range) proof of mode stability for Kerr-dS BHs**.
- The proofs of [5] makes use of **previously unknown symmetries of the point spectrum** of the radial Teukolsky equation conjectured by [1, 2] and proven by [5].
- The second proof of [5] is most useful in the case $s \in \mathbb{Z}$, where it rules out some of the modes in the superradiant range, where it is known from energy conservation that unstable and non-decaying modes, if they exist, must lie. [5]

New result from 2D CFT: greybody factor

- The emission spectrum as measured by an observer at infinity is no longer thermal, but is given by

$$\frac{\sigma(\omega)}{\exp\frac{\omega - m\Omega}{T_H} - 1} \tag{76}$$

  where $\sigma(\omega)$ is the so-called **greybody factor**, which can be defined as the flux going into the horizon normalized by the flux coming in from infinity $\sigma = \phi_{abs}/\phi_{in}$ [4].
- From the **solution of the connection problem** for the radial Teukolsky / confluent Heun equation, [4] obtain an exact expression of the **greybody factor** σ in terms of the instanton part of the **NS free energy** $\mathcal{F}^{inst}$.
  - Their result can be expressed in terms of only **BH parameters** by using **Matone's relation** (8).
  - They **checked their result with two different asymptotics**. One of them is the semiclassical result obtained by **WKB analysis of the ODE**, where it is given by the **dual SW period**

$$\sigma \simeq \exp\left\{-\frac{a_D}{\epsilon_1}\right\}, \qquad \epsilon_1 \to 0\,. \tag{77}$$

New result from 2D CFT: Love numbers

- Applying an **external gravitational field to a self-gravitating body** generically causes it to deform. The response of the body to the external gravitational tidal field is captured by the so-called **tidal response coefficients** or **Love numbers** [4].
- In general relativity, the tidal response coefficients are generally complex, and the **real part captures the conservative response** of the body, whereas the **imaginary part captures dissipative effects**. For four-dimensional Kerr black holes, the conservative (real part) of the response coefficient to static external perturbations has been found to vanish.
- Love numbers are **measurable quantities** that can be probed with **gravitational wave observations**. [4]
- **Using their 2D CFT approach** to the Teukolsky equation [4] compute
  - the **static** $\omega = 0$ **Love number** which agrees with the result in the literature;
  - the **non-static Love number** as an expansion in ω, that improves the result in the literature by also adding instanton corrections. [4]

Mathematical definition of quasinormal modes

Mathematical definition of quasinormal modes I

- Perturbations of the BH metric or fields can be shown to constitute solutions Φ of some PDE of the form
$$\left\{+\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+U(x)\right\}\Phi(t,x)=0\,, \tag{78}$$
where x here is a coordinate ("tortoise" coordinate) such that the BH horizon is put at $x\to-\infty$ and spacetime infinity at $x\to+\infty$.
- If we take the Laplace transform of Φ
$$\hat{\Psi}(s,x)=\int_0^\infty e^{-st}\Phi(t,x)\,dt\,, \tag{79}$$
$$\left\{-\frac{\partial^2}{\partial x^2}+U(x)+s^2\right\}\hat{\Psi}(s,x)=-\mathcal{I}(s,x)\,, \tag{80}$$
with $\mathcal{I}(s,x)=-s\Psi(t,x)\big|_{t=0}-\frac{\partial\Psi(t,x)}{\partial t}\big|_{t=0}$.

Mathematical definition of quasinormal modes II

- The corresponding homogeneous equation is exactly the ODE we are going to study in the next sections

$$\left\{-\frac{\partial^2}{\partial x^2} + U(x) + s^2\right\}\Psi(s,x) = 0\,. \tag{81}$$

Its solutions bounded at $x \to \pm\infty$, for $\mathrm{Re}\, s > 0$, are

$$\begin{aligned} \Psi_+(s,x) &\sim e^{-sx}\,, \qquad x \to +\infty \\ \Psi_-(s,x) &\sim e^{sx}\,, \qquad x \to -\infty\,. \end{aligned} \tag{82}$$

- The solution of the homogenous equation is then found to be given by the Green function G as

$$\hat{\Psi}(s,x) = \int_{-\infty}^{\infty} G(s,x,x')\mathcal{I}(s,x')dx'\,, \qquad G(s,x,x') = \frac{1}{W[\Psi_-,\Psi_+]}\Psi_-(s,x_<)\Psi_+(s,x_>)\,, \tag{83}$$

with $x_< = \min(x',x)$, $x_< = \max(x',x)$.

Mathematical definition of quasinormal modes III

- Then taking the antiplace transform of $\hat{\Psi}$

$$\Phi(t,x) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{st}\hat{\Psi}(s,x)\,ds\,, \tag{84}$$

we get the original perturbation as

$$\begin{aligned}\Phi(t,x) &= \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{st}\int_{-\infty}^{\infty} G(s,x,x')\mathcal{I}(s,x')\,dx'\,ds \\ &= \frac{1}{2\pi i}\oint e^{st}\frac{1}{W(s)}\int_{-\infty}^{\infty}\Psi_-(s,x_<)\Psi_+(s,x_>)\mathcal{I}(s,x')\,dx'\,ds \\ &= \sum_q e^{s_q t}\mathrm{Res}\left(\frac{1}{W(s)}\right)\bigg|_{s_q}\int_{-\infty}^{\infty}\Psi_-(s_q,x_<)\Psi_+(s_q,x_>)\mathcal{I}(s_q,x')\,dx'\,.\end{aligned} \tag{85}$$

We notice that the perturbation is written as a sum over the residues of the zeros of the wronskian of the fundamental regular solutions at $x \to \pm\infty$ (82):

Mathematical definition of quasinormal modes IV

- The crucial point for us is that the perturbation is a sum over the residues of the inverse wronskian of the regular solutions (82):

$$W(s_n) = W[\Psi_+, \Psi_-] = 0 \,. \tag{87}$$

  Besides, condition (87) means that at these special points the two solutions (82) (in general independent) become linearly dependent.

- By setting $s = i\omega$ we recover the usual intuitive definition of QNMs as the frequencies of plane wave solutions both incoming at the horizon and outgoing at infinity. However, as well explained in [Nollert:1999], this last definition is not mathematically rigorous, since it would lead to diverging boundary conditions. Instead, the QNMs ω_n have an imaginary part $\mathrm{Im}\,\omega_n < 0$, so that the perturbation they describe is damped to zero as $t \to \infty$.

A quantum integrable model for black holes

ODE for the perturbation of generalized RN BHs

- Line element for intersection of four stacks of D3-branes in type IIB supergravity:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]\,, \tag{88}$$

with

$$f(r) = \prod_{i=1}^{4}\left(1 + \frac{\mathcal{Q}_i}{r}\right)^{-\frac{1}{2}}\,. \tag{89}$$

- If the charges $\mathcal{Q}_i = Q = M$ are all equal, it leads to an extremal Reissner Nordström BH with $f(r) = 1 - \frac{2M}{r} + \frac{\mathcal{Q}^2}{r^2}$.
- The equation describing the scalar perturbation is

$$\frac{d^2\phi}{dr^2} + \left\{-\frac{(l+\frac{1}{2})^2 - \frac{1}{4}}{r^2} + \frac{\omega^2}{r^4}\left[\Sigma_4 + \Sigma_3 r + \Sigma_2 r^2 + \Sigma_1 r^3 + r^4\right]\right\}\phi = 0 \tag{90}$$

where we defined $\Sigma_k = \sum_{i_1<\ldots<i_k}^{4} \mathcal{Q}_{i_1}\cdots\mathcal{Q}_{i_k}$.
- It is a Doubly Confluent Heun equation with two irregular singularities at $r \to 0, \infty$.

ODE for generalized Perturbed Hairpin Integrable Model

- Changing variable and parameters as

$$r = \sqrt[4]{\Sigma_4}e^y \,, \qquad \omega\sqrt[4]{\Sigma_4} = -ie^\theta \,, \tag{91}$$

$$\frac{\Sigma_1}{\sqrt[4]{\Sigma_4}} = 2M_1 e^{-\theta} \,, \quad \frac{\Sigma_3}{\sqrt[4]{\Sigma_4^3}} = 2M_2 e^{-\theta} \,, \quad P^2 = (l+\frac{1}{2})^2 - \omega^2\Sigma_2 \,. \tag{92}$$

  the ODE takes the form

$$-\frac{d^2}{dy^2}\psi + \left[e^{2\theta}(e^{2y}+e^{-2y}) + 2e^\theta(M_1e^y + M_2e^{-y}) + P^2\right]\psi = 0 \,. \tag{93}$$

- In this form, this ODE is a generalization of the one for the Perturbed Hairpin Integrable Model:

$$-\frac{d^2}{dy^2}\psi + \left[e^{2\theta}(e^{2y}+e^{-ny}) + 2e^\theta Me^y + P^2\right]\psi = 0 \,. \tag{94}$$

Asymptotic solutions and discrete symmetries

- The regular solutions of (93) at $y \to \pm\infty$ ($j = 1, 2$) have boundary conditions

$$\begin{aligned} \psi_{-,0}(y) &\simeq 2^{-\frac{1}{2}-M_2} e^{-(\frac{1}{2}+M_2)\theta+(\frac{1}{2}+M_2)y} e^{-e^{\theta-y}} , \quad \mathrm{Re}\, y \to -\infty . \\ \psi_{+,0}(y) &\simeq 2^{-\frac{1}{2}-M_1} e^{-(\frac{1}{2}+M_1)\theta-(\frac{1}{2}+M_1)y} e^{-e^{\theta+y}} , \quad \mathrm{Re}\, y \to +\infty . \end{aligned} \tag{95}$$

- Equation (93) enjoys the discrete symmetries

$$\Omega_\pm : \ \theta \to \theta + i\pi/2 , \ y \to y \pm i\pi/2 , \ M_1 \to \mp M_1 , \ M_2 \to \pm M_2 , \tag{96}$$

- One can define other independent solutions as

$$\psi_{-,k} = \Omega_-^k \psi_{-,0} , \qquad \psi_{+,k} = \Omega_+^k \psi_{+,0} . \tag{97}$$

- We also have the invariance properties

$$\Omega_+^k \psi_{-,0} = \psi_{-,0} , \qquad \Omega_+^k \psi_{+,0} = \psi_{+,0} . \tag{98}$$

Baxter's Q function and quasinormal modes

- One can now define a Baxter's Q function as the wronskian

$$Q_{+,+}(\theta) = W[\psi_{+,0}, \psi_{-,0}] \,. \tag{99}$$

(We will use the notation $Q_{\pm,\pm} = Q(\theta, P, \pm M_1, \pm M_2)$, $Q_{\pm,\mp} = Q(\theta, P, \pm M_1, \mp M_2)$).

- Crucially, the QNMs condition (87) translates into

$$Q(\theta_n) = 0 \,, \tag{100}$$

namely the zeros of the Baxter's Q function which are the Bethe roots.

Central connection relation as QQ system

- The solutions are normalised such that their wronskians are $W[\psi_{-,k+1},\psi_{-,k}] = -i\exp\{(-1)^k i\pi M_2\}$, $W[\psi_{+,k+1},\psi_{+,k}] = i\exp\{(-1)^k i\pi M_1\}$.
- By the properties of wronskians $W[\psi_{\pm,1},\psi_{\pm,1}] = W[\psi_{\pm,0},\psi_{\pm,0}] = 0$, we can write the linear connection relations as

$$\begin{aligned} ie^{i\pi M_1}\psi_{-,0} &= Q_{-,+}(\theta + i\pi/2)\psi_{+,0} - Q_{+,+}(\theta)\psi_{+,1} \\ ie^{i\pi M_1}\psi_{-,1} &= Q_{-,-}(\theta + i\pi)\psi_{+,0} - Q_{+,-}(\theta + i\pi/2)\psi_{+,1} \end{aligned} \tag{101}$$

- Taking their wronskian we get the QQ system

$$Q_{+,-}(\theta + \frac{i\pi}{2})Q_{-,+}(\theta - \frac{i\pi}{2}) = e^{-i\pi(M_1-M_2)} + Q_{-,-}(\theta)Q_{+,+}(\theta)\,. \tag{102}$$

Q function as non-compact integral

- From the central connection relation (101) we get a formula for Q

$$Q_{+,+}(\theta) = -ie^{i\pi M_1} \lim_{y\to+\infty} \frac{\psi_{-,0}(y)}{\psi_{+,1}(y)} \tag{103}$$

$$= \int_{-\infty}^{\infty} dy \left[\sqrt{2\cosh(2y)}\Pi(y) - 2e^{\theta}\cosh y - \left(\frac{M_1}{1+e^{-y/2}} + \frac{M_2}{1+e^{y/2}}\right)\right] + \left(\theta + \frac{1}{2}\ln 2\right)(M_1 - M_2) \tag{104}$$

- The corresponding Riccati equation for $\Pi(y) = \frac{1}{\sqrt{2\cosh(2y)}}\frac{d}{dy}\ln(\sqrt[4]{-2\cosh(2y)}\psi(y))$ is

$$\Pi(y)^2 + \frac{1}{\sqrt{2\cosh(2y)}}\frac{d}{dy}\Pi(y)$$
$$= e^{2\theta} + e^{\theta}\mathrm{sech}(2y)\left(M_1 e^{y} + M_2 e^{-y}\right) + \frac{1}{2}\mathrm{sech}(2y)\left[P^2 + \mathrm{sech}(2y) - \frac{5}{4}\tanh^2(2y)\right] \tag{105}$$

Large energy expansion and Local integrals of motion

- For the $\theta \to +\infty$ expansion, setting

$$\Pi(y,\theta) \doteq e^{\theta} + \sum_{n=0}^{\infty} \Pi_n(y) e^{-n\theta} \qquad \theta \to +\infty \tag{106}$$

we get the recursion relation

$$\Pi_{n+1} = \frac{1}{2}\left(\frac{1}{\sqrt{2\cosh(2y)}}\frac{d}{dy}\Pi_n - \sum_{m=0}^{n} \Pi_m \Pi_{n-m}\right) \qquad n \geq 1 \tag{107}$$

with initial conditions $\Pi_0 = \frac{M_1 e^{y} + M_2 e^{-y}}{2\cosh(2y)}$, $\Pi_1 = \frac{1}{2}\left(\frac{1}{\sqrt{2\cosh(2y)}}\frac{d}{dy}\Pi_0 - \Pi_0^2 - U\right)$.

- The (vaccum eigenvalues) local integrals of motion (LIMs) are given by the integrals

$$C_n I_n(P, M_1, M_2) = \int_{-\infty}^{\infty} dy\, \sqrt{-2\cosh(2y)}\Pi_n(y, P, M_1, M_2)$$

$$Q(\theta, P, M_1, M_2) \doteq \exp\left\{-\frac{4\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2} e^{\theta} + (\theta + \frac{1}{2}\ln 2)(M_1 - M_2) + \sum_{n=1}^{\infty} e^{-\theta n} C_n I_n\right\} \qquad \theta \to +\infty \tag{108}$$

Y function and quasinormal modes

- One can define a Y function as

$$Y_{+,+}(\theta) = e^{i\pi(M_1-M_2)} Q_{+,+}(\theta) Q_{-,-}(\theta) \tag{109}$$

- From the QQ system it follows the Y system as

$$Y_{+,-}(\theta + \frac{i\pi}{2}) Y_{-,+}(\theta - \frac{i\pi}{2}) = [1 + Y_{+,+}(\theta)][1 + Y_{-,-}(\theta)]\,. \tag{110}$$

- Eventually, the QQ system written as

$$e^{i\pi(M_1+M_2)} Q_{+,+}(\theta + \frac{i\pi}{2}) Q_{-,-}(\theta - \frac{i\pi}{2}) = 1 + Y_{+,-}(\theta)\,. \tag{111}$$

  characterizes the QNMs as

$$Y_{+,-}(\theta_n - i\pi/2) = -1 \qquad Y_{-,+}(\theta_n + i\pi/2) = -1 \tag{112}$$

Thermodynamic Bethe Ansatz

- It can be inverted in a Thermodynamic Bethe Ansatz (TBA) for $\varepsilon_{\pm,\pm}(\theta) = -\ln Y_{\pm,\pm}(\theta)$:

$$\varepsilon_{\pm,\pm}(\theta) = \frac{8\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2} e^{\theta} \mp i\pi(M_1 - M_2) - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{\ln[1+\exp(-\varepsilon_{\pm,\mp}(\theta'))] + \ln[1+\exp(-\varepsilon_{\mp,\pm}(\theta'))]}{\cosh(\theta-\theta')}$$
$$\varepsilon_{\pm,\mp}(\theta) = \frac{8\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2} e^{\theta} \mp i\pi(M_1 + M_2) - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{\ln[1+\exp(-\varepsilon_{\pm,\pm}(\theta'))] + \ln[1+\exp(-\varepsilon_{\mp,\mp}(\theta'))]}{\cosh(\theta-\theta')} \tag{113}$$

- P enters as the $\theta \to -\infty$ boundary condition (from perturbative expansion of ODE)

$$\varepsilon_{+,+}(\theta, P) \sim 4P\theta + 2C(P, M_1, M_2)\,, \qquad \theta \to -\infty \tag{114}$$

$$C(P, M_1, M_2) = \ln\left[\frac{2^{1-2P} P\,\Gamma(2P)^2}{\sqrt{\Gamma\left(P+\frac{1}{2}-M_1\right)\Gamma\left(P+\frac{1}{2}+M_1\right)\Gamma\left(P+\frac{1}{2}-M_2\right)\Gamma\left(P+\frac{1}{2}-M_2\right)}}\right] \tag{115}$$

Gravity Y system and TBA

- In gravity variables the Y system reads

$$
\begin{aligned}
&Y(\theta+\frac{i\pi}{2},-i\Sigma_1,-\Sigma_2,i\Sigma_3,\Sigma_4)Y(\theta-\frac{i\pi}{2},-i\Sigma_1,-\Sigma_2,i\Sigma_3,\Sigma_4)\\
&=[1+Y(\theta,\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4)][1+Y(\theta,-\Sigma_1,\Sigma_2,-\Sigma_3,\Sigma_4)]
\end{aligned}
\tag{116}
$$

- It can be inverted into the TBA

$$
\begin{aligned}
\varepsilon_{\pm,\pm}(\theta)&=[f_{0,+}\mp\frac{i\pi}{2}(\frac{\Sigma_1}{\Sigma_4^{1/4}}-\frac{\Sigma_3}{\Sigma_4^{3/4}})]e^{\theta}-\varphi*(\bar{L}_{\pm\pm}+\bar{L}_{\mp\mp})(\theta)\\
\bar{\varepsilon}_{\pm,\pm}(\theta)&=[\bar{f}_{0,+}\pm\frac{\pi}{2}(\frac{\Sigma_1}{\Sigma_4^{1/4}}+\frac{\Sigma_3}{\Sigma_4^{3/4}})]e^{\theta}-\varphi*(L_{\pm\pm}+L_{\mp\mp})(\theta)
\end{aligned}
\tag{117}
$$

where we defined $\varepsilon(\theta)=-\ln Y(\theta,\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4)$, $\bar{\varepsilon}(\theta)=\varepsilon(\theta,i\Sigma_1,-\Sigma_2,-i\Sigma_3,\Sigma_4)$, $L=\ln[1+\exp\{-\varepsilon\}]$, $\varphi(\theta)=(2\pi\cosh(\theta))^{-1}$, $f_{0,+}=c_{0,+,+}+c_{0,-,-}$ and $c_{0,\pm,\pm}=\int_{-\infty}^{\infty}\left[\sqrt{2\cosh(2y)\pm\frac{\Sigma_1}{\sqrt[4]{\Sigma_4}}e^{y}\pm\frac{\Sigma_3}{\sqrt[4]{\Sigma_4^3}}e^{-y}+\frac{\Sigma_2}{\sqrt{\Sigma_4}}}-\text{reg.}\right]dy$ and boundary condition $\varepsilon_{\pm,\pm}(\theta)\sim 4(l+\frac{1}{2})\theta+2C(P,0,0)$, $\theta\to-\infty$.

Quasinormal modes from TBA

- The QNMs condition in gravity variables reads alternatively as

$$\bar{Y}_{+,+}(\theta_{n'} - i\pi/2) = -1\,, \quad Q_{+,+}(\theta_n) = 0 \tag{118}$$

or

$$\bar{\varepsilon}_{+,+}(\theta_{n'} - i\pi/2) = -i\pi(2n' + 1)\,. \tag{119}$$

- Through the quantization condition on $\bar{\varepsilon}_{+,+}$ we can actually numerically compute QNMs from TBA.

n	l	TBA	Leaver	WKB
0	1	$\underline{0.869}623 - \underline{0.372}022i$	$\underline{0.869}32 - \underline{0.372}859i$	$0.89642 - 0.36596i$
0	2	$\underline{1.477}990 - \underline{0.368}144i$	$\underline{1.477}888 - \underline{0.368}240i$	$1.4940 - 0.36596i$
0	3	$\underline{2.080}200 - \underline{0.367}076i$	$\underline{2.080}168 - \underline{0.367}097i$	$2.0916 - 0.36596i$
0	4	$\underline{2.680}363 - \underline{0.366}637i$	$\underline{2.680}350 - \underline{0.366}642i$	$2.6893 - 0.36596i$

Table: Comparison of QNMs in different methods ($n' = 0$, $\Sigma_1 = \Sigma_3 = 0.2$, $\Sigma_2 = 0.4$, $\Sigma_4 = 1$).

Explanation of connection to $\mathcal{N} = 2$ gauge theory

Classical Seiberg-Witten curve and differential

- Given an operator $H(\hat{x},\hat{p})$, with $[\hat{x},\hat{p}]=i\hbar$, we associate it with a classical curve and a one form

$$H(x,p)=u\,,\qquad \lambda(x,u)=f(p(x,u))\,d\,x \tag{120}$$

- We consider operators $H(\hat{x},\hat{p})$ such that the corresponding classical curve (120) coincides with a SW curve of a suitable 4D $\mathcal{N}=2$ gauge theory.
- In particular we will consider $SU(2)$ $N_f=2$ gauge theory, with SW curve and SW differential:

$$p^2=x^3-x^2u-\frac{\Lambda_2^4}{64}(x-u)+\frac{\Lambda_2^2}{4}m_1m_2x-\frac{\Lambda_2^4}{64}(m_1^2+m_2^2)\,. \tag{121}$$

$$\lambda=-\frac{\sqrt{2}}{4\pi}\frac{dx}{p}\left[x-u-\frac{\Lambda_2^2}{16}\frac{(m_1-m_2)^2}{x-\frac{\Lambda_2^2}{8}}+\frac{\Lambda_2^2}{16}\frac{(m_1+m_2)^2}{x+\frac{\Lambda_2^2}{8}}\right]\,. \tag{122}$$

Classical Seiberg-Witten periods

- If the curve (120) has genus g, we choose a basis of cycles $\mathcal{A}_i, \mathcal{B}_i$, with $i=1,\ldots,g$ and define classical periods by integrating over such cycles

$$\Pi_{A_i,B_i}(u) = \oint_{\mathcal{A}_i,\mathcal{B}_i} \lambda(x,u)\, dx\,, \qquad i=1,\ldots,g\,. \tag{123}$$

- (123) are the SW periods and they encode the central charges and masses of the BPS particles in the theory.
- For $SU(2)$ $N_f=2$:

$$\int_{A,B}\lambda = \frac{\sqrt{2}}{4\pi}\left[\frac{4}{3}uI_1^{A,B} - 2I_2^{A,B} + \frac{\Lambda_2^2}{8}(m_1-m_2)^2 I_3^{A,B}\left(\frac{\Lambda_2^2}{8}-\frac{u}{3}\right) - \frac{\Lambda_2^2}{8}(m_1+m_2)^2 I_3^{A,B}\left(-\frac{\Lambda_2^2}{8}-\frac{u}{3}\right)\right] \tag{124}$$

The integral is taken over different cycles A and B between the branch points e_1, e_2, e_3 of the SW curve p. I_1, I_2, I_3 are combinations of elliptic integrals of 1st, 2nd and 3rd kind.

Quantum Seiberg-Witten curve and integrability dictionary

- Quantum Seiberg Witten curve in the Nekrasov-Shatashvili limit of the Ω-background (used to compute instanton contributions) $\epsilon_2 \to 0, \epsilon_1 = \hbar$ with $x \sim e^{-y}$

$$H_{N_f}\psi(y) = u\psi(y) \tag{125}$$

$$H_{N_f} = -\hbar^2 \frac{d^2}{dy^2} + \frac{1}{2}\Lambda_{N_f}^{2-N_f/2}\left[e^y K_+\left(\hbar\frac{d}{dy} - u_1 + \frac{\hbar}{2}\right) + e^{-y}K_-\left(\hbar\frac{d}{dy} - u_1 - \frac{\hbar}{2}\right)\right] - u_0 \ . \tag{126}$$

- For $N_f = 2 = (1,1)$ we have

$$-\hbar^2\frac{d^2}{dy^2}\psi + \left[\frac{\Lambda_2^2}{8}\cosh(2y) + \frac{1}{2}m_1\Lambda_2 e^y + \frac{1}{2}m_2\Lambda_2 e^{-y} + u\right]\psi = 0 \tag{127}$$

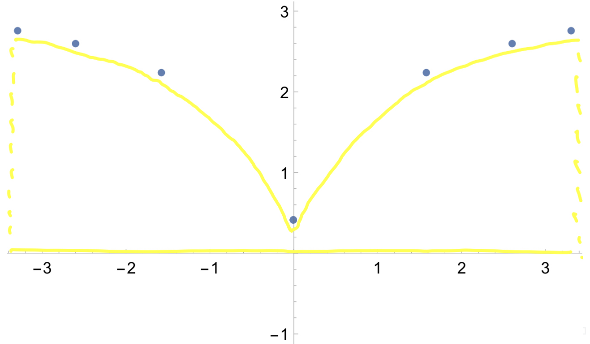
- We get this equation from the Generalized Perturbed Hairpin Integrable model ODE (93) through the dictionary

$$\frac{\hbar}{\Lambda_2} = \frac{1}{4}e^{-\theta}\,, \qquad \frac{u}{\Lambda_2^2} = \frac{1}{16}P^2 e^{-2\theta}\,, \qquad \frac{m_1}{\Lambda_2} = \frac{1}{4}M_1 e^{-\theta}\,, \qquad \frac{m_2}{\Lambda_2} = \frac{1}{4}M_2 e^{-\theta}\,, \tag{128}$$

Quantum SW periods I

- We can define exact periods by exact integrals of $\mathcal{P}(y) = -i\frac{d}{dy}\ln\psi(y)$ as sums over residues at the poles which as $\hbar \to 0$ reduce to the classical cycles (branch cuts).

$$\Pi_{A,B}(u,\hbar) \doteq \oint_{A,B} \mathcal{P}(\hbar,y,u)\,dy = 2\pi i \sum_n \mathrm{Res}\mathcal{P}(y)\Big|_{y_n^{A,B}} \qquad \Pi_A = a\,, \quad \Pi_B = a_D \quad (129)$$



Quantum SW periods II

- Alternatively, we can define
  - the B period through the Nekrasov-Shatashvili free energy $\mathcal{F}^{(N_f)}(a, \mathbf{m}, \Lambda_{N_f}, \hbar)$, which can be computed systematically through combinatorial calculus on the gauge group representation:

$$A_D(u, \mathbf{m}, \Lambda_{N_f}, \hbar) = \Pi_B^{inst}(u, \mathbf{m}, \Lambda_{N_f}, \hbar) = \partial_a \mathcal{F}(u, \mathbf{m}, \Lambda_{N_f}, \hbar)\Big|_{a=a(u,\mathbf{m},\Lambda_{N_f},\hbar)} \tag{130}$$

  - the A period $\Pi_A = a$ from Matone's relation:

$$u = a^2 - \frac{\Lambda_{N_f}}{4 - N_f} \frac{\partial \mathcal{F}_{\text{inst}}^{(N_f)}(a; \mathbf{m}, \Lambda_{N_f}, \hbar)}{\partial \Lambda_{N_f}} \tag{131}$$

Gauge TBA

- In gauge variables the Y system reads

$$
\begin{aligned}
& Y(\theta+\frac{i\pi}{2},-u,-im_1,im_2,\Lambda_2)Y(\theta-\frac{i\pi}{2},-u,-im_1,im_2,\Lambda_2) \\
& =[1+Y(\theta,u,m_1,m_2,\Lambda_2)][1+Y(\theta,u,-m_1,-m_2,\Lambda_2)]
\end{aligned}
\tag{132}
$$

so it can be inverted in the TBA similar to but different from the gravity one (133)

$$
\begin{aligned}
\varepsilon_{\pm,\pm}(\theta) &= [f_{0,+}\mp\frac{4\pi i}{\Lambda_2}(m_1-m_2)]e^{\theta}-\varphi*(\bar{L}_{\pm\pm}+\bar{L}_{\mp\mp})(\theta) \\
\bar{\varepsilon}_{\pm,\pm}(\theta) &= [\bar{f}_{0,+}\mp\frac{4\pi}{\Lambda_2}(m_1+m_2)]e^{\theta}-\varphi*(L_{\pm\pm}+L_{\mp\mp})(\theta)
\end{aligned}
\tag{133}
$$

where we defined $\varepsilon(\theta)=-\ln Y(\theta,u,m_1,m_2,\Lambda_2)$, $L=\ln[1+\exp\{-\varepsilon\}]$ and in particular $\bar{\varepsilon}(\theta)=\varepsilon(\theta,-u,-im_1,+im_2,\Lambda_2)$. The kernel is still $\varphi(\theta)=\frac{1}{2\pi}\frac{1}{\cosh\theta}$. The forcing term is given by $f_{0,\pm}=-c_{0,+,\pm}-c_{0,-,\mp}$ with $c_{0,\pm,\pm}=\int_{-\infty}^{\infty}\left[\sqrt{2\cosh(2y)\pm\frac{8m_1}{\Lambda_2}e^{y}\pm\frac{8m_2}{\Lambda_2}e^{-y}+\frac{16u}{\Lambda_2^2}}-\text{reg.}\right]dy$.

Quantum gauge B period from TBA

- We find that the $\theta \to \infty$ asymptotic expansion for $\varepsilon = \sum_{n=0}^{\infty} e^{\theta(1-2n)} \varepsilon^{(n)}$ gives actually the gauge periods $\hbar \to 0$ asymptotic expansion $a_D \doteq \sum_{n=0}^{\infty} \hbar^{2n} a_D^{(n)}$ coefficients as computed from the ODE (WKB approximation).

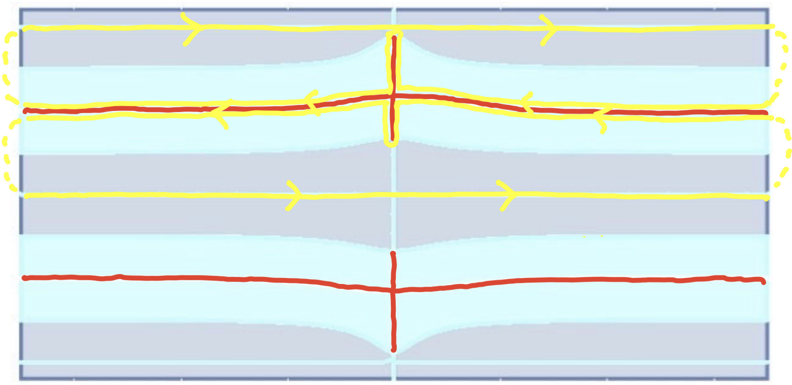
$$\begin{aligned} \bar{\varepsilon}^{(n)}_{+,+} &= \varepsilon^{(n)}(-u, im_1, -im_2, \Lambda_2) = (-1)^n \frac{8\sqrt{2}\pi}{\Lambda_2} a_D^{(n)}(u, m_1, m_2, \Lambda_2) \\ \bar{\varepsilon}^{(n)}_{-,-} &= \varepsilon^{(n)}(-u, -im_1, im_2, \Lambda_2) = (-1)^n \frac{8\sqrt{2}\pi}{\Lambda_2} a_D^{(n)}(u, m_1, m_2, \Lambda_2) + \delta_{n,0} \frac{8\pi}{\Lambda_2}(m_1 + m_2) \end{aligned} \tag{134}$$

- Also considering the exact integrals we can prove

$$\begin{aligned} \varepsilon(\theta, -u, im_1, -im_2, \Lambda_2) &= \frac{8\sqrt{2}\pi}{\Lambda_2} a_D(\hbar, u, m_1, m_2, \Lambda_2) \\ \varepsilon(\theta, -u, -im_1, im_2, \Lambda_2) &= \frac{8\sqrt{2}\pi}{\Lambda_2} a_D(\hbar, u, m_1, m_2, \Lambda_2) + \frac{8\pi}{\Lambda_2}(m_1 + m_2) e^{\theta} \end{aligned} \tag{135}$$

Quantum gauge B period as pseudoenergy: geometrical proof

- We can prove analytically the equality of the pseudonergy and gauge period by relating the different integration contours in the complex plane (in white the branch cuts).



Quasinormal modes from gauge periods quantization conditions

- We see that from our formalism it follows immediately that QNMs are given by quantization conditions on the gauge integral periods

$$\frac{8\sqrt{2}\pi}{\Lambda_2} a_D(\hbar, u, m_1, m_2, \Lambda_2) = -i\pi(2n' + 1) \tag{136}$$

- Recently other authors found QNMs to be given by quantization conditions on the gauge instanton periods

$$A_D(\hbar, u, m_1, m_2, \Lambda_2) = i\pi n \tag{137}$$

- In the limit to $N_f = 0$ $SU(2)$ we know a consistent relation between them

$$Q(\theta, P) = \exp\left\{\frac{2\pi i}{\hbar} a_D(u, \Lambda_0, \hbar)\right\} = i\frac{\sinh \frac{1}{\hbar} A_D(a, \Lambda_0, \hbar)}{\sinh \frac{2\pi i}{\hbar} a}, \tag{138}$$

(also $N_f = 1$ another similar.)

Baxter's T functions and TQ relations

- Now, the presence of the irregular singularity of (93) at $y \to \pm\infty$ (Stokes phenomenon) plays a rôle for defining the T functions

$$T_{+,+}(\theta) = -iW[\psi_{-,-1}, \psi_{-,1}]\,, \quad \tilde{T}_{+,+}(\theta) = iW[\psi_{+,-1}, \psi_{+,1}]\,. \tag{139}$$

By expanding $\psi_{\pm,1}$ in terms of $\psi_{\pm,0}$, $\psi_{\pm,-1}$ as

$$\psi_{+,-1} = -e^{-2\pi i M_1}\psi_{+,1} + e^{-i\pi M_1}\tilde{T}_{+,+}(\theta)\psi_{+,0} \tag{140}$$

$$\tilde{T}_{+,+}(\theta)\psi_{+,0} = e^{i\pi M_1}\psi_{+,-1} + e^{-i\pi M_1}\psi_{+,+1} \tag{141}$$

we obtain the TQ relations

$$\begin{aligned} T_{+,+}(\theta)Q_{+,+}(\theta) &= e^{i\pi M_2}Q_{+,-}(\theta - \frac{i\pi}{2}) + e^{-i\pi M_2}Q_{+,-}(\theta + \frac{i\pi}{2}) \\ \tilde{T}_{+,+}(\theta)Q_{+,+}(\theta) &= e^{i\pi M_1}Q_{-,+}(\theta - \frac{i\pi}{2}) + e^{-i\pi M_1}Q_{-,+}(\theta + \frac{i\pi}{2})\,. \end{aligned} \tag{142}$$

- By the invariance under the symmetries $\Omega_\pm$ also the periodicities follow

$$T_{-,+}(\theta + i\pi/2) = T_{+,+}(\theta)\,, \quad \tilde{T}_{+,-}(\theta + i\pi/2) = \tilde{T}_{+,+}(\theta) \tag{143}$$

Baxter's T function, Floquet exponent and gauge A period

- One can prove a relation between the T function for the doubly confluent Heun equation and its Floquet exponent (such that $\psi(y+2\pi i)=e^{2\pi i\nu}\psi(y)$)

$$\begin{aligned} 2\cos 2\pi\nu + 2\cos 2\pi M_1 &= \tilde{T}_{+,+}(\theta)\,\tilde{T}_{-,+}(\theta+i\frac{\pi}{2}) \\ 2\cos 2\pi\nu + 2\cos 2\pi M_2 &= T_{+,+}(\theta)\,T_{+,-}(\theta+i\frac{\pi}{2})\,. \end{aligned} \tag{144}$$

- For the $SU(2)$ $N_f=0$ gauge theory by definition

$$\nu = \Pi_A = a \tag{145}$$

and therefore another fundamental link between integrability and gauge theory.

Proof of alternative quantization condition for quasinormal modes for $N_f = 0$

- From the relation between T and a it follows a proof of the alternative quantization condition for quasinormal modes for $N_f = 0$

$$a(\theta_n) = \frac{n}{2}\,, \qquad n \in \mathbb{Z}\,. \tag{146}$$

which was given by some authors for the D3 brane (corresponding to $SU(2)$ $N_f = 0$).

- Indeed, the TQ relation $T(\theta)Q(\theta) = Q(\theta - i\pi/2) + Q(\theta + i\pi/2)$ means $Q(\theta_n - i\pi/2) + Q(\theta_n + i\pi/2) = 0$. This and the QQ relation $Q(\theta + i\pi/2)Q(\theta - i\pi/2) = 1 + Q^2(\theta)$ actually fixes $Q(\theta_n + i\pi/2)Q(\theta_n - i\pi/2) = 1$ and then

$$Q(\theta_n \pm i\pi/2) = \pm i \tag{147}$$

are fixed, too. Again the QQ relation around θ_n forces $Q(\theta + i\pi/2) = i \pm Q(\theta) + \dots$ and $Q(\theta - i\pi/2) = -i \pm Q(\theta) + \dots$ up to smaller corrections (dots). Therefore, TQ relation imposes

$$T(\theta_n) = 2\cos 2\pi a(\theta_n) = \pm 2\,. \tag{148}$$

Partial generalization of alternative quantization condition for quasinormal modes

- By making considerations on these TQ systems and the QQ system (111) like in section III, we are not in general able to conclude that the T-s are quantized, except in the case of equal masses $M_1 = M_2$ where we find

$$T_{+,+}(\theta_n)\,T_{-,-}(\theta_n) = 4\,, \qquad (M_1 = M_2)\,. \tag{149}$$

  that generalizes (148).

- It corresponds to $(M_1 = M_2 = M)$

$$\begin{gathered} T_{+,+}(\theta_n)\,T_{-,-}(\theta_n) = 4 \\ [\cos 2\pi\nu + \cos 2\pi M]_{\theta=\theta_n} = \pm 2\,. \end{gathered} \tag{150}$$

Interpretation of integrability duality

- Moreover $\psi_{+,0}(y) = \psi_{-,0}(-y)$ is just the symmetry that exchange infinity ($y \to +\infty$) and the (analogue) horizon ($y \to -\infty$), leaving the photon sphere ($y = 0$) fixed as in [BianchiRusso:2021] (thanks to identifications of certain scattering angles with the SW period). In this respect, under this symmetry we have

$$\tilde{T}_{+,+}(\theta) = T_{+,+}(\theta) \qquad (M_1 = M_2) \tag{151}$$

Comparison of methods of computation of quasinormal modes

Comparison of methods of computation of quasinormal modes

n	l	TBA	Leaver	WKB
0	1	$0.896681 - 0.40069i$	N.A.	$0.93069 - 0.39458i$
0	2	$1.5308 - 0.39676i$	N.A.	$1.5511 - 0.39458i$
0	3	$2.15708 - 0.395689i$	N.A.	$2.1716 - 0.39458i$
0	4	$2.78077 - 0.39525i$	N.A.	$2.7921 - 0.39458i$

Table: Here $\Sigma_1 \neq \Sigma_3$ and the Leaver method seems not applicable (N.A.).

- Computing QNMs numerically has been until now typically quite laborious and this has been also due to the difficulties in developing exact analytic characterizations of QNMs.
- The standard analytic method is the one with the continuous fractions by Leaver. However, it requires that the ODE is such that its solution be expressed in power series with coefficients satisfying a 3-point recursion. Anyway, there is a recent development, the so-called matrix Leaver method, which is still applicable.
- A new analytic characterization appeared very recently as QNMs have been obtained by Nekrasov partition functions of $\mathcal{N} = 2$ supersymmetric gauge, *i.e.* theories. Nevertheless, we found it very difficult to reach, by summing instantons, the QNMs values $|\Lambda_{N_f}|/\hbar \gg 1$. It needs a clever numerical re-summation procedure.

Generalization and perspectives

Present limitations and future generalizations

- What we have described holds for the generalizazion of RN BHs (intersection of 4 stacks of D3 branes). It corresponds to $SU(2)$ $N_f = 2, 1, 0$ gauge theories.
- However, from the generality of construction it is manifest that our method applies to many other theories. We can list, for instance, the $SU(2)$ $N_f = (2, 0)$ gauge theory and the associated gravity counterparts, like D1D5 fuzzballs and CCLP 5D BHs, as well as to $SU(2)$ $N_f = 3$ gauge theory and the general asymptotically flat Kerr-Newman BH. The associated integrable structure appears to be a generalization of the conformal minimal models. In these cases the (confluent Heun) ODE has one irregular and two regular singulatities.
- Another simple but tricky case happens when all the (Heun) ODE singularities are regular, e.g. $SU(2)$ $N_f = 4$ or $SU(2)$ quivers that is asymptotically AdS BHs. The corresponding IM appear to be spin chains.
- Analytic continuations of the TBAs in θ and the moduli are necessary to obtain overtones ω_n $n \geq 1$ and some particular gravitational systems.

Perspectives

- For AdS BHs the 3-fold correspendence Integrability/Gauge/Gravity could become 4-fold through holographic duality to CFT.
- Besides generalization, it is very intriguing to study the application of the integrability structure beyond the determination of the QNMs for the study of gravitational solutions.
- Eventually, we note that much of the BH theory seems to go in parallel to the ODE/IM correspondence construction and its 2D statistical field theory interpretation, beyond the determination of QNMs: as an example, the *greybody factor* or *absorption coefficient* seems a ratio of Qs.

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Xièxie!

Thank you!