AdS black holes from the ODE/IM correspondence perspective

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AdS BHs and perturbation ODEs

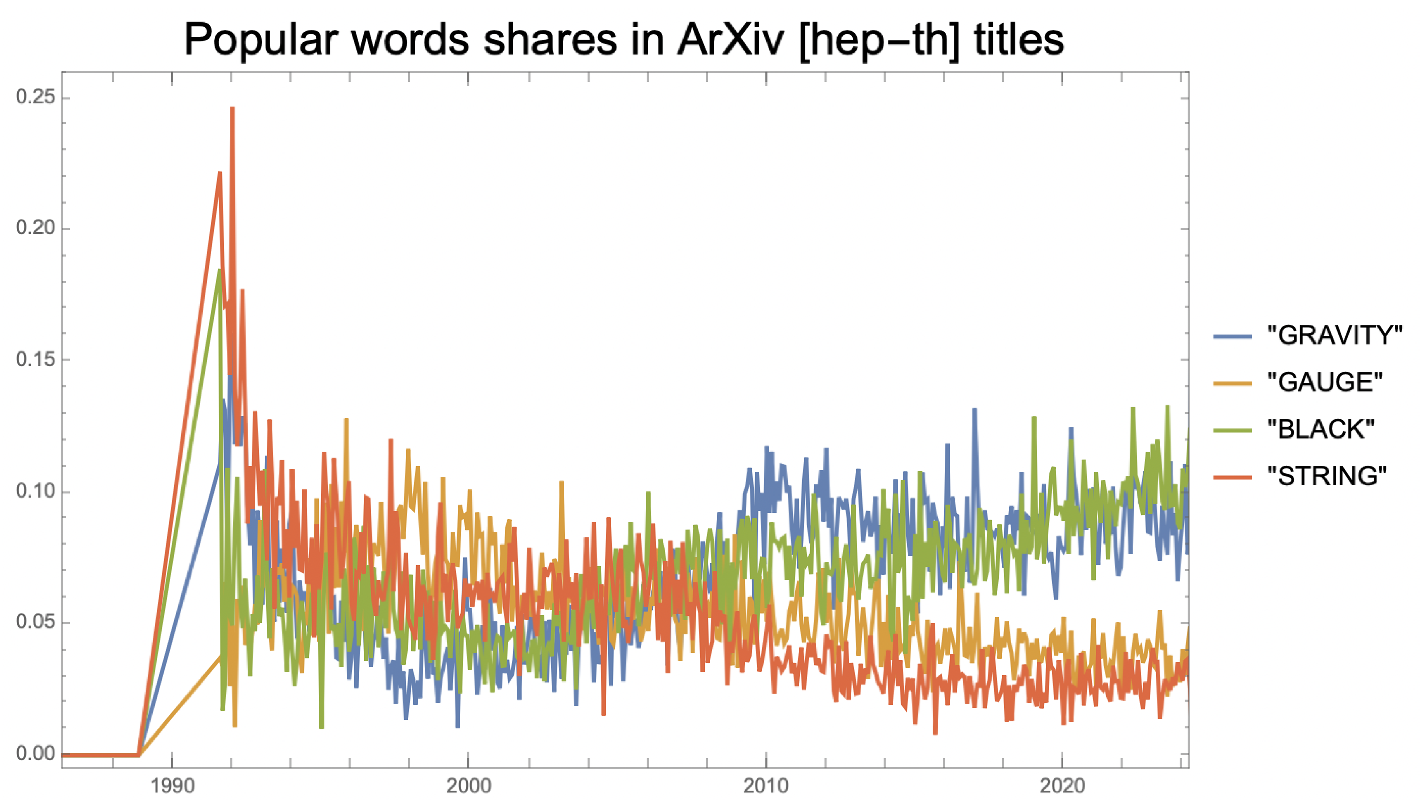
An **AdS black hole (BH)** is obtained, for example, as the near horizon limit $r \ll R$ of the D3 brane background:

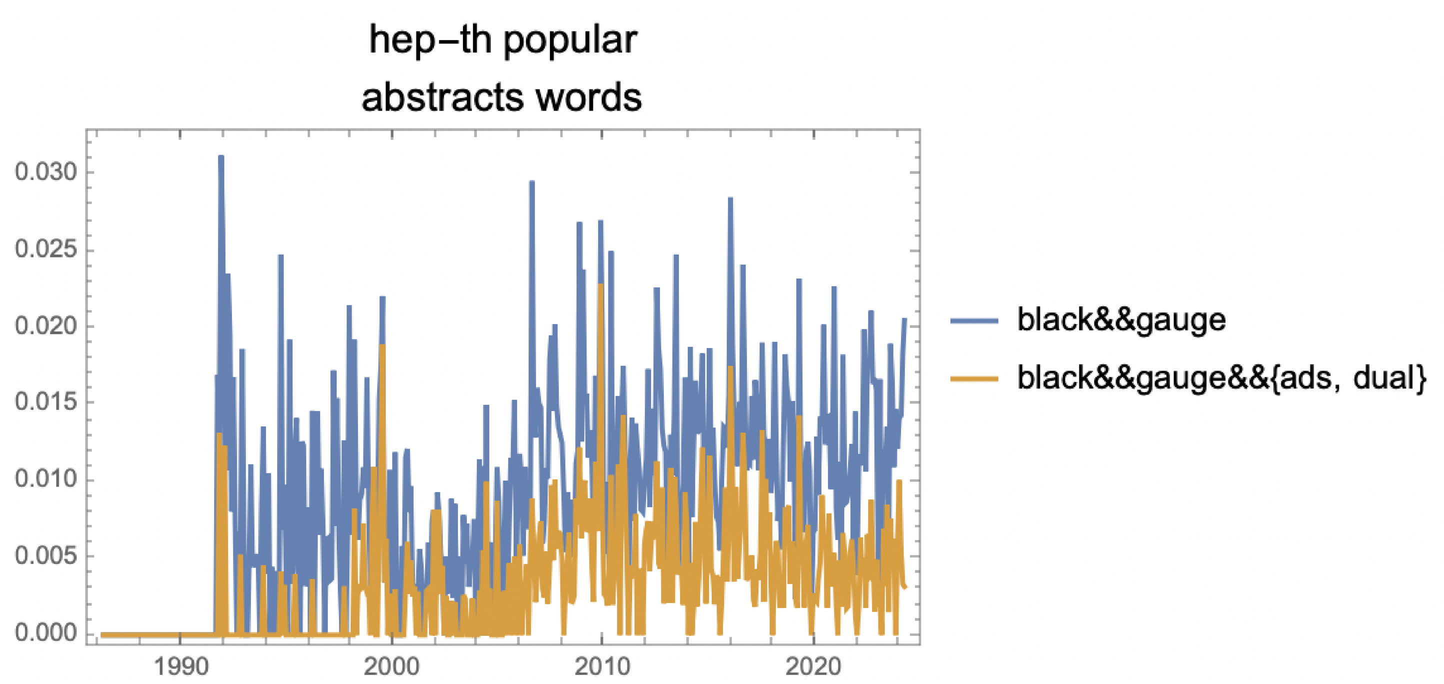
$$ds^2_{10} = \frac{(\pi T R)^2}{u}[-f(u)dt^2 + dx^2 + dy^2 + dz^2] + \frac{R^2}{4u^2 f(u)}du^2 + R^2 d\Omega_5^2\,, \tag{1}$$

where $u = r_0^2/r^2$ and $f(u) = 1 - u^2$. Dropping the angular term, we obtain a five-dimensional unperturbed metric $g^{(0)}_{\mu\nu}$ on which we can consider perturbations $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$. The **Einstein equations**, for example, for the **perturbation** $\phi = h^x_y = uh_{xy}/(\pi T R)^2$ reduce to a simple **Fuchsian ODE**:

$$\phi'' - \frac{1+u^2}{uf}\phi' + \frac{\omega^2 - q^2 f}{uf^2}\phi = 0\,. \tag{2}$$

(AdS) BHs on ArXiv





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Fuchsian ODE/IM correspondence

In all cases of AdS BHs, perturbations are governed by ODEs of the **Fuchsian** type, that is with only **regular singularities**. For example, **hypergeometric** ODE for AdS_3 or **Heun** ODE for AdS_5, namely

$$w''(z) + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t}\right)w'(z) + \frac{(\alpha\beta z - q)}{z(z-1)(z-t)}w(z) = 0\,, \tag{3}$$

In ODE/IM correspondence, the **Baxter's Q functions** are directly defined as Wronskians of fundamental solutions

$$Q_{ij} \propto W[w_i, w_j]\,. \tag{4}$$

The fundamental Heun solutions change character or are left invariant, under action of the **discrete symmetries** of the ODE

$$\Omega_i w_i = v_i \to \infty\,, \qquad \Omega_j w_i = w_i \to 0\,, \qquad z \to z_i\,. \tag{5}$$

Starting from the linear (central) **connection relations**, it is possible to derive 6 different QQ **systems** aka **quantum Wronskian equations**:

$$Q_{ij}\,[\Omega_i\Omega_j Q_{ij}] - [\Omega_i Q_{ij}]\,[\Omega_j Q_{ij}] = 1\,. \tag{6}$$

In this Fuchsian case, because of the absence of irregular singularities and associated **Stokes phenomenon**, it is not immediately obvious how to define T **and** Y **functions**...
Nevertheless, some interesting progress is being made [1, 5].

IM counterparts: spin chains?

We should stress that ODE/IM correspondence allows derivation and use of the **integrability structures directly from the ODE**, in principle even **without knowing a priori the corresponding IM**. However, in some particular cases one could prove a correspondence with some **spin chain**:

- a particular case of Heun equation, with polynomial solutions, is the **generalized Pöeschl-teller** equation, in correspondence with spin $L/2 - 1$ **XXZ spin chain**, with $\Delta = \cos \pi/L$ [3];
- a limit of Heun equation is the **hypergeometric** equation and that again is in correspondence with **XXZ spin chain**, at the **supersymmetric point** $\Delta = -1/2$ [5].

We should notice though that, while spin chains are **discrete** systems, BHs parameters are naturally **continuous**.

Solving the quantum Wronskian equations

Through some sophisticated ODE/IM analysis, we can express the **exact solution** of the QQ systems (6) in terms of the $\mathcal{N} = 2\; SU(2)$ **gauge theory prepotential** $\mathcal{F}$:

$$Q_{1,2} \propto \exp\left\{\sum_{k=1}^{N_f}\frac{\partial\mathcal{F}}{\partial m_k}\right\}\frac{\sinh\left\{\frac{\partial\mathcal{F}}{\partial a}\right\}}{\sin 2\pi a}\,, \tag{7}$$

$$Q_{k,l} \propto \exp\left\{\frac{\partial\mathcal{F}}{\partial m_k} + \frac{\partial\mathcal{F}}{\partial m_l}\right\}\,. \tag{8}$$

The latter in gauge theory can be **computed perturbatively**, either in the **instanton weak coupling** $\Lambda \to 0$ regime, or in the $\hbar \to 0$ **asymptotic semiclassical** expansion (exactly in Λ).
From (7) it follows a closed formula for the CFT_4 **holographic Green function**:

$$G_R \propto \frac{\Omega_1 Q_{1,2}}{Q_{1,2}}\,. \tag{9}$$

Other authors derived the latter through **AGT duality**, but we were able to avoid it and use **purely the mathematics of ODE/IM**. In perspective, ODE/IM could also provide a **properly exact** method of computation for G_R, overcoming the need for perturbative expansions of $\mathcal{F}$: the **Thermodynamic Bethe Ansatz (TBA)** [2, 4].

References

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