New applications of Supersymmetry and Integrability to Black Holes' Ringdown

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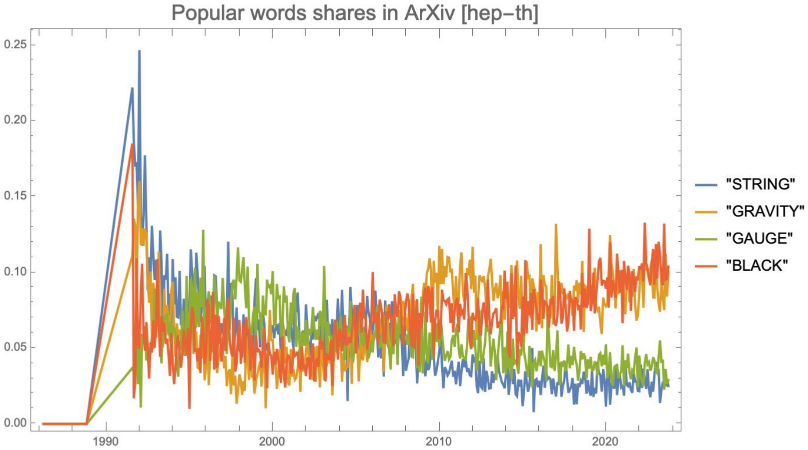
Based on: arXiv:2311.*****, arXiv:2208.14031, arXiv:2112.11434, arXiv:1908.08030

with D. Fioravanti, R. Mahanta, M. Rossi, H. Shu

On the high energy theory "stock market"

- Counting title words of **ALL 161000+ arXiv[hep-th] papers**:

1	theory	17661
2	quantum	15865
3	gravity	11848
4	field	11709
5	black	11590
6	model	9448
7	gauge	8536
8	theories	8496
9	n	8483
10	string	7251



- Interesting **temporal trends (1986/4 - 2023/10)** appear for the most popular article **title words' shares** (= number of occurences/ total number of articles, per month) ...

ArXiv [hep-th] data analysis

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   Partition[Drop[KeyValueMap[#1 → Normal[#2] &, wordCountMonthsHepthUpTo[35](*keep this large for accuracy*)], 7], UpTo[12]]] /. colorReplacements //
  TableForm
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1992	quantum → 251	string → 192	gravity → 155	field → 135	2 → 107	1 → 90	n → 88	black → 85
1993	quantum → 396	string → 215	field → 209	2 → 186	gravity → 174	n → 156	gauge → 149	black → 144
1994	quantum → 476	field → 241	string → 230	2 → 208	gravity → 177	gauge → 167	n → 145	black → 143
1995	quantum → 373	string → 279	field → 274	gauge → 249	2 → 249	n → 214	gravity → 167	black → 161
1996	quantum → 370	2 → 307	n → 296	field → 274	gauge → 264	string → 243	black → 202	1 → 162
1997	quantum → 362	n → 310	gauge → 299	2 → 275	string → 260	field → 259	black → 205	supersymmetric → 171
1998	quantum → 383	gauge → 327	n → 321	field → 317	string → 215	black → 203	2 → 203	supersymmetric → 125
1999	quantum → 371	field → 330	gauge → 312	n → 282	string → 247	2 → 223	black → 182	gravity → 163
2000	field → 376	quantum → 363	gauge → 328	string → 261	n → 250	noncommut. → 240	brane → 192	2 → 187
2001	quantum → 403	field → 352	gauge → 282	string → 276	n → 253	noncommut. → 235	2 → 228	brane → 200
2002	quantum → 412	field → 360	gauge → 301	string → 296	n → 278	2 → 214	black → 197	gravity → 194
2003	quantum → 426	field → 369	string → 287	gauge → 286	n → 284	black → 223	gravity → 201	2 → 193
2004	quantum → 435	field → 375	n → 301	black → 288	string → 285	gauge → 263	gravity → 218	2 → 211
2005	quantum → 425	black → 310	field → 309	string → 293	gauge → 266	n → 247	gravity → 236	energy → 176
2006	quantum → 446	black → 337	field → 332	string → 301	gravity → 301	gauge → 294	n → 245	2 → 171
2007	quantum → 464	black → 364	gravity → 327	field → 326	string → 295	gauge → 295	n → 265	holes → 195
2008	quantum → 399	gravity → 356	black → 341	n → 330	field → 305	gauge → 272	string → 234	holes → 192
2009	gravity → 484	quantum → 434	black → 399	field → 396	n → 329	gauge → 286	string → 218	holes → 218
2010	gravity → 581	quantum → 480	black → 439	field → 383	gauge → 300	n → 281	holes → 258	holographic → 235
2011	gravity → 534	quantum → 505	black → 422	field → 420	gauge → 311	n → 308	holographic → 245	holes → 225
2012	gravity → 519	quantum → 509	black → 418	field → 412	n → 317	gauge → 304	holographic → 251	holes → 207
2013	gravity → 511	quantum → 483	black → 428	field → 404	gauge → 289	n → 272	holographic → 248	holes → 231
2014	gravity → 555	quantum → 532	field → 428	black → 417	inflation → 329	n → 311	gauge → 263	holes → 226
2015	quantum → 567	gravity → 531	black → 489	field → 446	n → 279	holes → 264	gauge → 260	inflation → 252
2016	quantum → 581	gravity → 513	black → 487	field → 452	holographic → 288	n → 287	gauge → 279	holes → 266
2017	quantum → 613	gravity → 556	black → 480	field → 435	gauge → 257	holes → 249	n → 243	holographic → 231
2018	quantum → 634	gravity → 560	black → 510	field → 479	n → 285	gauge → 262	2 → 261	holographic → 253
2019	quantum → 690	black → 641	gravity → 550	field → 485	holes → 337	n → 323	hole → 279	gauge → 273
2020	quantum → 764	gravity → 689	black → 685	field → 474	holes → 387	n → 306	2 → 284	hole → 279
2021	quantum → 786	black → 664	gravity → 603	field → 481	holes → 392	gauge → 268	2 → 263	n → 257
2022	quantum → 815	black → 745	gravity → 690	field → 485	holes → 392	hole → 312	gravitational → 275	gauge → 273
2023	quantum → 679	black → 603	gravity → 508	field → 354	holes → 323	hole → 261	gravitational → 216	gauge → 206

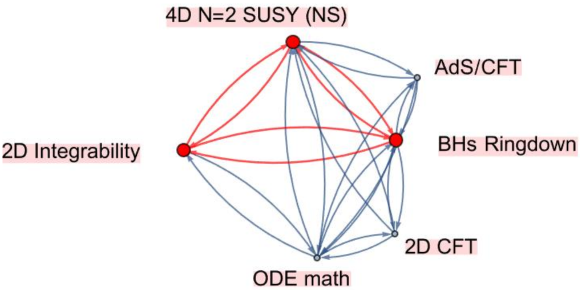
Check out the **notebook** of my little arXiv[hep-th] data analysis [work in progress]:

`https://github.com/Daniele-Gregori/ArXiv-Hepth-Data-Analysis`

How to **interpret** the **general shift** away from "string" / "gauge" towards "gravity" / "black"?

Outline

1. From supersymmetry to gravitational waves
   - $\mathcal{N} = 2$ SUSY primer
   - ODEs for black holes
   - A new gauge/gravity duality
   - Applications and related fields
2. Integrability for black holes
   - Introduction to integrability
   - Introduction to ODE/IM
   - Black holes perturbation theory
   - QNMs as Bethe roots
   - Other applications
3. Solving quantum Wronskian via $\mathcal{N} = 2$ SUSY
   - Exact closed formulae for Q
   - On new gauge/gravity duality
   - Comparison of QNMs comp. methods
4. Conclusions



From supersymmetry to gravitational waves

Summary of $\mathcal{N} = 2$ gauge (Seiberg-Witten) theory

- **Seiberg-Witten (SW) theory** is the effective theory for $\mathcal{N} = 2$ SUSY gauge theory. [Seiberg-Witten, hep-th/9407087, 9408099]
- Its lagrangian is expressed in terms of ($\mathcal{N} = 1$) chiral ϕ and vector W_α superfields and the **SW prepotential** $\mathcal{F}^{(0)}$

$$\mathcal{L}_{SW} = \frac{1}{16\pi}\mathrm{Im}\int d^4x \left[\frac{1}{2}\int d^2\theta\, \mathcal{F}^{(0)''}(\phi) W^\alpha W_\alpha + \int d^2\theta d^2\bar{\theta}\phi^\dagger \mathcal{F}^{(0)'}(\phi)\right] . \qquad (1)$$

- $\mathcal{F}^{(0)}$ can be obtained from **classical SW periods** $a^{(0)}$, $a_D^{(0)}$, which are monodromies, or period integrals, of the SW differential $\lambda \sim \sqrt{p^2}$ of a hyperelliptic curve $p^2 = x^3 - x^2u + (...)x + (...)$.

Quantum Seiberg-Witten theory

- To compute **instanton contributions**, it is convenient to deform spacetime - becoming the Ω-**background** - in terms of two parameters ϵ_1, ϵ_2.
  [Nekrasov, hep-th/0206161]
- We will consider the **Nekrasov-Shatashvili (NS) limit** $\epsilon_2 \to 0$, in which the prepotential is obtained from monodromies of **quantum SW curves** in $y \sim -\ln x$

$$-\hbar^2 \frac{d^2}{dy^2}\psi(y) + [U(y;\mathbf{m},\Lambda) - u]\,\psi(y) = 0\,, \tag{2}$$

  as ODEs of Schrödinger time-independent type with $\epsilon_1 = \hbar \neq 0$, for $SU(2)$ gauge group.
  [Nekrasov-Shatashvili, arXiv:0908.4052]
- The SW prepotential is recovered as leading $\hbar \to 0$ leading asymptotic of the **NS prepotential** $\mathcal{F}$

$$\mathcal{F} = \frac{1}{\hbar}\mathcal{F}^{(0)} + O(\hbar) + O(e^{-\frac{1}{\hbar}})\,, \qquad \hbar \to 0\,. \tag{3}$$

Nekrasov-Shatashvili instanton series

- The NS prepotential can be obtained as a **series in the instanton coupling** Λ

$$\mathcal{F}(\Lambda, a, \hbar) = \mathcal{F}^{pert}(a, \hbar) + \sum_{n=1}^{\infty} \mathcal{F}_n^{inst}(a, \hbar)\Lambda^{(4-N_f)n}\,. \tag{4}$$

- The **gauge period** a is recovered from the **ODE moduli parameter** u through

$$u = \Lambda \frac{\partial \mathcal{F}(\Lambda, a, \hbar)}{\partial \Lambda}\,. \tag{5}$$

- Also the **gauge dual instanton period** A_D is defined as

$$A_D = \frac{\partial \mathcal{F}(\Lambda, a, \hbar)}{\partial a} \tag{6}$$

and in the $\hbar \to 0$ SW limit, $A_D \simeq \frac{1}{\hbar} A_D^{(0)}$, with $A_D^{(0)} = a_D^{(0)} + 2\pi i a^{(0)}$.

Example of instanton series

- For example for $SU(2)$ gauge theory with $N_f = 3$ matter flavours, at **second order** we get:

$$
\begin{aligned}
\mathcal{F}^{inst}(\Lambda_3, m_1, m_2, m_3, a, \hbar) = &-\frac{m_1 m_2 m_3}{4(4a^2-\hbar^2)}\Lambda_3 \\
&+\left[\frac{-256a^8 - 256a^6\left(m_3^2+m_1^2+m_2^2-2\hbar^2\right)+96a^4\left(2m_3^2\hbar^2+2m_2^2\left(4m_3^2+\hbar^2\right)+2m_1^2\left(4m_3^2+4m_2^2+\hbar^2\right)-3\hbar^4\right)}{8192\left(a^2-\hbar^2\right)\left(4a^2-\hbar^2\right)^3}\right. \\
&+\frac{-16a^2\left(3m_3^2\hbar^4+3m_2^2\left(8m_3^2\hbar^2+\hbar^4\right)+m_1^2\left(8m_2^2\left(10m_3^2+3\hbar^2\right)+3\left(8m_3^2\hbar^2+\hbar^4\right)\right)-4\hbar^6\right)}{8192\left(a^2-\hbar^2\right)\left(4a^2-\hbar^2\right)^3} \\
&\left.+\frac{4m_3^2\hbar^6+4m_2^2\hbar^6+48m_3^2m_2^2\hbar^4+4m_1^2\hbar^2\left(12m_3^2\hbar^2-4m_2^2\left(28m_3^2-3\hbar^2\right)+\hbar^4\right)-5\hbar^8}{8192\left(a^2-\hbar^2\right)\left(4a^2-\hbar^2\right)^3}\right]\Lambda_3^2+O(\Lambda_3^3)\,. \qquad (7)
\end{aligned}
$$

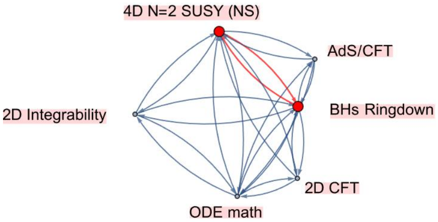
- The computation algorithm is based on **combinatorics on Young diagrams**

Nice analytical expressions...but the terms are infinite and the computation time for higher orders grows exponentially!

No jokes: Schwarshild black hole!

- The quantum SW curve for $N_f = 3$ $SU(2)$ **gauge th.** can be mapped to the Regge-Wheeler equation for linear **perturbations of the Schwarschild BH!**

$$
\begin{aligned}
&-\hbar^2 \frac{d^2}{dy^2}\psi(y) + \Big\{ e^{2y}\Lambda_3 \left(4(m_1 - m_2)^2\right) \\
&+ \left(\Lambda_3^2 - 24\Lambda_3 m_3 + 64u\right) + 4e^{y}\sqrt{\Lambda_3}\left(-2\hbar^2 + 8m_1 m_2 + \Lambda_3 m_3 - 8u\right) \\
&+ 4e^{-y}\sqrt{\Lambda_3}\left(8m_3 - \Lambda_3\right) + 4\Lambda_3 e^{-2y}\Big\} \frac{1}{16\left(\sqrt{\Lambda_3}e^{y} - 2\right)^2}\psi(y) = 0
\end{aligned}
\tag{8}
$$



$$\Lambda_3 = -16iM\omega\,, \qquad u = l(l+1) - 8M^2\omega^2 + \frac{1}{4}\,, \qquad \Big\Downarrow \qquad r = \frac{4M}{\Lambda_3}e^{-y}\,,$$

$$\hbar = 1\,, \qquad m_1 = s - 2iM\omega\,, \qquad m_2 = -s - 2iM\omega\,, \qquad m_3 = -2iM\omega\,,$$

$$\left[f(r)\frac{d}{dr}f(r)\frac{d}{dr} + \omega^2 - f(r)\left[\frac{l(l+1)}{r^2} + (1-s^2)\frac{2M}{r^3}\right]\right]\phi(r) = 0\,, \tag{9}$$

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \quad \text{with } f(r) = 1 - 2M/r$$

Heun equations and black holes

Generally in both **General Relativity** and **modified gravity**, BHs perturbations obey different kind of ODEs.

- Perturbations of **asymptotically AdS$_5$ BHs** obey **Heun** equation

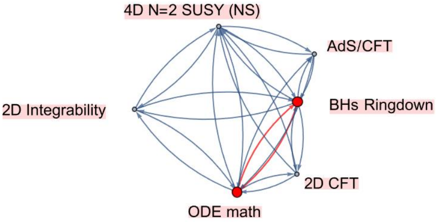
$$w''(z) + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\alpha+\beta-\gamma-\delta+1}{z-a}\right) w'(z) + \frac{\alpha\beta z - q}{z(z-1)(z-a)} w(z) = 0\,. \qquad (10)$$

- for 4D **asymptotically flat BHs**, **Confluent Heun** equation (CHE)

$$w''(z) + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon\right) w'(z) + \frac{\alpha z - q}{z(z-1)} w(z) = 0\,. \qquad (11)$$

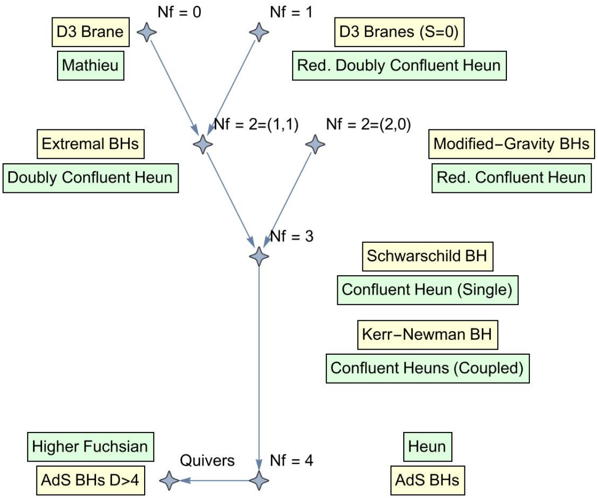
- for 4D **Extremal** flat BHs, **Doubly Confluent Heun** eq. (DCHE)

$$w''(z) + \left(\frac{\delta}{z^2} + \frac{\gamma}{z} + 1\right) w'(z) + \frac{\alpha z - q}{z^2} w(z) = 0\,. \qquad (12)$$

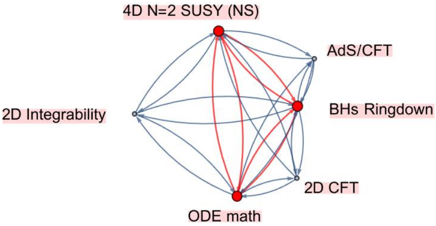


- Caveat: **rotating BHs** involve **coupled ODEs** (separable PDEs)!

General picture of $\mathcal{N} = 2$ gauge - gravity correspondence



$SU(2)$ $\mathcal{N} = 2$ (NS) gauge theories, Heun ODEs and BHs models.



[Aminov-Grassi-Hatsuda, arXiv:2006.06111, Bianchi-Consoli-Grillo-Morales, arXiv:2109.09804]

A new $\mathcal{N} = 2$ gauge - gravity duality

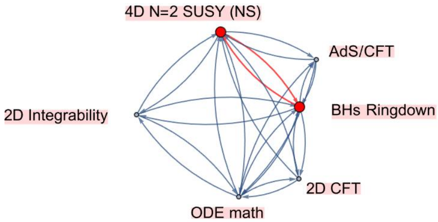
- It was found heuristically in [Aminov-Grassi-Hatsuda, arXiv:2006.06111] that **quasinormal modes (QNMs) are given by quantization conditions on the gauge periods**

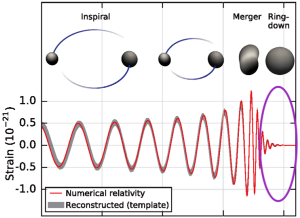
$$A_D(\Lambda, \mathbf{m}, u) = 2i\pi\hbar\, n \quad \text{for } n \in \mathbb{N} \tag{13}$$

$$\Downarrow$$

$$\text{QNMs} \quad \omega_n \propto \Lambda$$

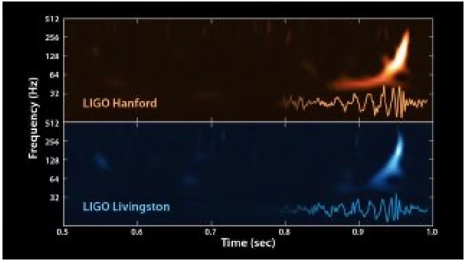
- QNMs can be observed as the complex frequencies of the **damped oscillations of spacetime** in the final - **ringdown** - phase of black holes (BHs) merging. [Nollert, Class.Quant.Grav. 16 (1999); Berti-Cardoso-Starinets, arXiv:0905.2975]





Importance for modified gravity and phenomenology

- Even if QNMs in GR are well understood, it is still an open problem to compute them in **modified gravity** theories, so developing **new analytic characterizations and numerical methods** is very important.
- This could constitute a mathematical empowerment of **gravitational phenomenology** and **search of new physics**. [Cardoso-Pani, arXiv:1709.01525]



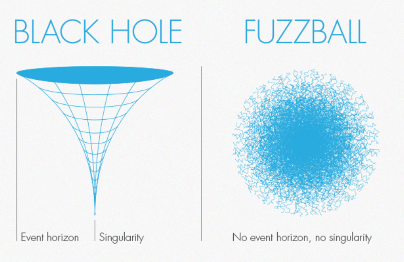
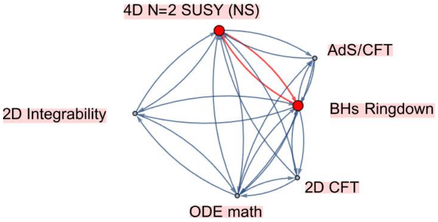


Figure: GW150914 (LIGO), Quanta Magazine

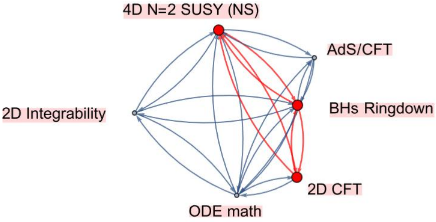
Improved QNMs computations and theorems

- $\mathcal{N} = 2$ gauge theory was recently applied to **effectively compute QNMs** of many GR and modified gravity BHs. [Bianchi-Consoli-Grillo-Morales, arXiv:2105.04245, 2109.09804, ...]
- Also theoretical results:
  - an **isospectral simpler equation** to the Kerr perturbation ODE [Hatsuda, arXiv:2007.07906];
  - improved **theoretical proofs of BHs stability** [Casals-Costa, arXiv:2105.13329; Bianchi-Benedetto-Russo-Sudano, arXiv:2305.00865; Bianchi-Russo-Grillo-Morales-Sudano, arXiv:2305.15105];
  - a **simpler interpretation of Chandrasekhar transformation** as exchange of gauge mass parameters [Nakajima-Lin, arXiv:2111.05857];
  - precise determination of the **conditions of invariance under Couch-Torrence transformations** [Bianchi-Di Russo, arXiv:2110.09579, 2203.14900].



Connection to 2D CFT and AGT duality

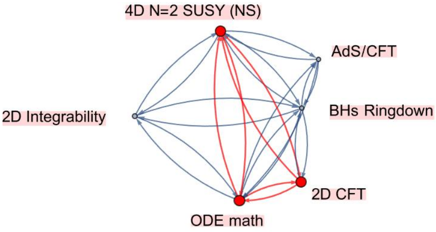
- **AGT duality** identifies the 4D $\mathcal{N} = 2$ gauge partition function with the 2D Liouville CFT conformal blocks, essentially solutions of qSW curves identified with BPZ ODEs.
  [Alday-Gaiotto-Tachikawa, arXiv:0906.3219]
- Thanks to this connection, **further results for BHs physics** followed
  [Bonelli-Iossa-Lichtig-Tanzini, arXiv:2105.04483]
  - **more accurate computations** of observable quantities such as **Love numbers** - describing tidal deformations - and **greybody factors** - for Hawking radiation.
  - a **proposal of "physical explanation"** of the new $\mathcal{N} = 2$ gauge - QNMs correspondence.



We will show later our alternative conceptually simpler explanation.

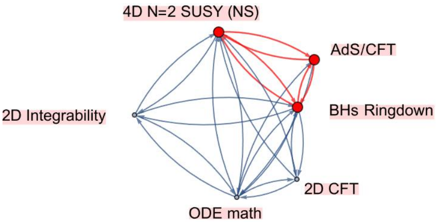
Connection to mathematics of Heun equations

- Again from AGT duality it was possible to derive closed formulas for connection coefficients of Heun equations.
  [Bonelli-Iossa-Lichtig-Tanzini, arXiv:2201.04491]
- Which is allowing further developments...
  [Lisovyy-Naidiuk, arXiv:2208.01604;
  Kwon-Amado, arXiv:2307.12824;
  ... see below]



Connection to AdS/CFT

- Applying $\mathcal{N} = 2$ gauge theory to asymptotically AdS_5 BHs - $SU(2)$ $N_f = 4$ NS theory - it was dervied an **exact formula** for the thermal scalar **two-point function** in 4D **holographic CFT** [Dodelson-Grassi-Iossa-Lichtig-Zhiboedov, arXiv:2206.07720]
- similar results in [Bhatta-Mandal, arXiv:2211.02449; Giusto-Iossa-Russo, arXiv:2306.15305; Aminov-Arnaudo-Bonelli-Grassi-Tanzini, arXiv:2307.10141; He-Li, arXiv:2308.13518; Lei-Shu-Yang-Zhang-Zhu, arXiv:2308.16677]



Integrability for black holes

On quantum integrability

- Integrability can be considered as the study of **non-linear** phenomena in nature in a quantitative **exact, non perturbative** way.
- The hallmark of quantum integrability is the presence of **infinite integrals of motion commuting** with each other

$$[\mathbf{I}_n, \mathbf{I}_m] = 0 \,. \tag{14}$$

What objects are more general and abstract?

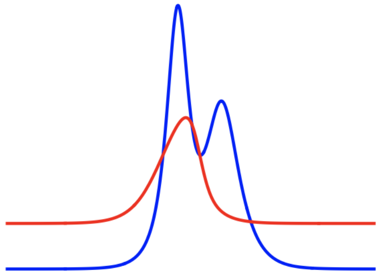


Figure: Solitons of mKdV (Wolfram Demo)

- The IM I_{2n-1} are also asymptotic expansion coefficients, for large spectral parameter $\lambda = e^{\theta}$, of the **Baxter's** Q, T **operators**

$$\ln \mathbf{Q}(\theta) \simeq -C_0 e^{\theta} - \sum_{n=1}^{\infty} e^{-n\theta} C_n \mathbf{I}_n \qquad \theta \to +\infty\,, \tag{15}$$

$$[\mathbf{Q}(\theta), \mathbf{T}(\theta)] = 0\,, \tag{16}$$

[Bazhanov-Lukyanov-Zamolodchikov, hep-th/9412229, hep-th/9604044]

- Their eigenvalues, the Q, T **functions** satisfy **functional relations**, like **quantum wronskian** QQ or **Baxter's** TQ (here for Sinh-Gordon IM)

$$Q(\theta - i\pi/2)Q(\theta + i\pi/2) - Q(\theta - i\pi a/2)Q(\theta + i\pi a/2) = 1\,, \tag{17}$$

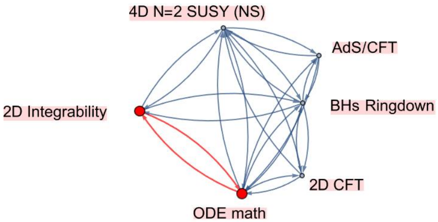
$$T(\theta)Q(\theta) = Q(\theta + i\pi(a+1)/2) + Q(\theta - i\pi(a+1)/2)\,. \tag{18}$$

- Given suitable **asymptotic behaviours** these functional relations have **unique solutions**.

The ODE/IM correspondence

- The **ODE/IM correspondence** is an approach to integrability, in which the Q, T functions and all the (QQ, TQ...) **integrable structures** of some Integrable Model (IM) can be **derived from an Ordinary Differental Equation** (ODE), even without knowing a priori the corresponding IM. [Dorey-Tateo, hep-th/9812211, Bazhanov-Lukyanov-Zamolodchikov, hep-th/9812247, Dorey-Dunning-Tateo, hep-th/0703066]

ODE	IM
Spectral determinant	Q
Stokes multipliers	T_k
Central Connect. rels.	QQ relation
Lateral Connect. rels.	TQ, T-syst
ODE parameters	IM parameters
ODE ind. var. x	... cf. below



ODE/IM allows to apply integrability to very different physical theories!

ODE/IM correspondence essentials

- The first ODE/IM instance found was for the **anharmonic oscillator ODE**, with $\alpha \geq -1$

$$-\frac{d^2}{dx^2}\psi(x) + \left(x^{2\alpha} + \frac{l(l+1)}{x^2} - E\right)\psi(x) = 0\,. \tag{19}$$

- The **spectral determinants** are also computed as **Wronskians**

$$Q^+(E,l) = Q_0^+ \prod_{k=0}^{\infty}\left(1 - \frac{E}{E_{2k}}\right) = W[\psi_{+,0}, \psi_{-,0}] \text{ with } \begin{cases} \psi_{+,0}(x) \to 0\,, & x \to \infty \\ \psi_{-,0}(x) \to 0\,, & x \to 0 \end{cases}. \tag{20}$$

- $Q^+(E,l)$ can be **computed** as **integral** of the solution $\mathcal{P}(x) = -i\frac{d}{dx}\ln\psi(x)$ of the **Riccati** ODE equivalent to (19)

$$Q^+(E,l) = \lim_{x\to 0}[x^l\psi_{+,0}(x)] = \exp\left\{ i\int_0^{\infty}[\mathcal{P}(x) - \text{regulariz.}\,]\,dx\right\}\,. \tag{21}$$

- Through **discrete symmetries** of the ODE, we can define other solutions in $\mathbb{C}$, with $\omega = e^{\frac{i\pi}{\alpha+1}}$

$$\Omega_+ : x \to \omega^{-1} x\,, \quad E \to \omega^2 E \quad \Longrightarrow \quad \psi_{+,k} \equiv \Omega_+^k \psi_{+,0} = \psi_{+,0}(\omega^{-k} x, \omega^{2k} E, l)\,, \tag{22}$$

so we can write **lateral connection relations** in terms **Stokes multipliers** $T_{k/2}$

$$\psi_{-1}(x) = T_{k/2}(E,l)\psi_{k-1}(x) + \tilde{T}_{k/2}(E,l)\psi_k(x) \quad \Longrightarrow \quad T_{k/2}(E,l) = W[\psi_{-1}, \psi_k]\,. \tag{23}$$

- Similarly defining $\psi_{-,1}(x,E,l) \equiv \psi_{-,0}(x,E,-l-1)$ and $Q^-(E,l) = Q^+(E,-l-1)$ from **central connection relations** (between $\psi_{-,k}$ and $\psi_{+,k}$) we obtain QQ **system**

$$\omega^{1/2} Q^+(\omega^{-1}E) Q^-(\omega E) - \omega^{-1/2} Q^+(\omega E) Q^-(\omega^{-1} E) = 2i\,. \tag{24}$$

- Similarly also TQ and T systems.
- $Q^\pm$, $T_{k/2}$ realize **vacuum** Q, T **functions** of **Scaling 6-vertex IM!** [Dorey-Tateo, hep-th/9812211, Bazhanov-Lukyanov-Zamolodchikov, hep-th/9812247]

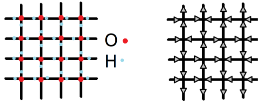


Figure: (Wikicommons/Moali)

Other ODE/IM examples: Liouville

- We can reach the regime $\alpha < -1$, by a formal change of variable, leading to the ODE for the to **Liouville model** , that is CFT central charge $c = 1 + (b + 1/b)^2 \geq 1$

$$-\frac{d^2}{dy^2}\psi(y) + \left[e^{2\theta}(e^{y/b} + e^{-by}) + P^2\right]\psi(y) = 0\,, \tag{25}$$

  which reduces to (modified) Mathieu equation for $b = 1$ ($c = 25$).
  [Fioravanti-DG, arXiv:1908.08030]

- Liouville model is
  - the **2D quantum gravity** theory
  - dual to 4D $\mathcal{N} = 2$ SUSY through **AGT duality**

We make a different connection through ODE/IM!

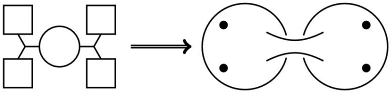


Figure: AGT duality quiver (NLab)

Other ODE/IM examples: Perturbed Hairpin, Paperclip

- **Perturbed Hairpin IM** (for $n = 1, 2$) corresponds to Doubly Confluent Heun eq.

$$-\frac{d^2}{dx^2}\psi(x) + \left[2kqe^x + k^2(e^{2x} + e^{-nx}) + P^2\right]\psi(x) = 0\,. \tag{26}$$

- **Paperclip IM** (for $n = 2$) corresponds to the Confluent Heun eq.

$$-\frac{d^2}{dx^2}\psi(x) + \left[k^2\,(e^x+1)^n - \frac{p^2 e^x}{e^x+1} - \frac{\left(q^2-\frac{1}{4}\right)e^x}{(e^x+1)^2}\right]\psi(x) = 0\,. \tag{27}$$

- They are both related **2D IQFTs** with **non-conformal boundary interaction**. [Fateev-Lukyanov, hep-th/0510271; Lukyanov-Vitchev-Zamolodchikov, hep-th/0312168]

Fig. 2. The paperclip formed by a junction of two hairpins.

On the origin of ODE/IM

- For a long time, ODE/IM has been regarded as either **mysterious correspondence** or just a mathematical coincidence.
- However, recent work [Fioravanti-Rossi, arXiv:2106.07600] has shed light on its **origin**, by "reversing it":

QQ/TQ/Bethe Ansatz

$\Downarrow$

Marchenko-like int. eq.

$\Downarrow$

Schrödinger ODEs

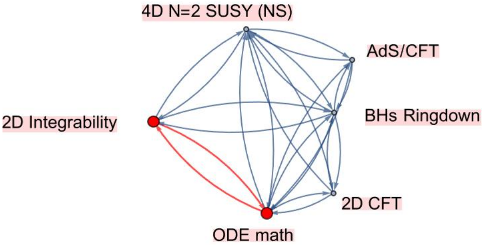


Figure: The Thinker (Rodin)

Three different physical theories, same mathematics!

$$-\frac{d^2}{dy^2}\psi(y) + \left[2e^{2\theta}\cosh y + P^2\right]\psi(y) = 0 \qquad \text{INTEGRABILITY (sd-Liouville)} \tag{28}$$

$$\frac{\hbar}{\Lambda_0} = \frac{\epsilon_1}{\Lambda_0} = e^{-\theta} \qquad \frac{u}{\Lambda_0^2} = \frac{1}{2}P^2 e^{-2\theta} \qquad \Downarrow$$

$$-\frac{\hbar^2}{2}\frac{d^2}{dy^2}\psi(y) + [\Lambda^2\cosh y + u]\psi(y) = 0 \qquad \text{N=2 GAUGE TH. } (SU(2)\ N_f = 0) \tag{29}$$

$$r = Le^{y/2} \quad \omega L = -2ie^{\theta} \quad P = \frac{1}{2}(l+2) \qquad \Downarrow \qquad \phi(r) = e^{y/2}\psi(y)$$

$$\frac{d^2\phi}{dr^2} + \left[\omega^2\left(1+\frac{L^4}{r^4}\right) - \frac{(l+2)^2 - \frac{1}{4}}{r^2}\right]\phi(r) = 0 \qquad \text{BLACK HOLES PERT. (D3 brane)} \tag{30}$$

- We have shown this **triality** generally for Mathieu , Doubly Confluent Heun , Confluent Heun ODEs .

Black holes perturbation theory

- For perturbations of the **Schwarzschild BH** metric $g^{(0)}_{\mu\nu}$, **Einstein equations**, after **linearization** and **gauge fixing** reduce to just a couple of much simpler PDEs:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{31}$$

$$\Big\Downarrow \quad h_{\mu\nu} = g_{\mu\nu} - g^{(0)}_{\mu\nu} \Rightarrow (Z_l, Q_l)$$

1) for the **axial** perturbations (phase $(-1)^{l+1}$), the **Regge-Wheeler** equation

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right)\right] Q_l(t,x) = 0\,; \tag{32}$$

2) for the **polar** perturbations (phase $(-1)^{l}$), the **Zerilli** equation

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \left(1 - \frac{2M}{r}\right)\left(\frac{72M^3}{r^5\lambda(r)^2} + (\dots)\frac{12M}{r^3\lambda(r)^2}\left(1 - \frac{3M}{r}\right) + (\dots)\frac{1}{r^2\lambda(r)}\right)\right] Z_l(t,x) = 0, \tag{33}$$

where $\lambda(r) = l(l+1) - 2 + 6M/r$ and $r = r(x)$, with x tortoise coordinate.

From PDEs to ODEs

- The **Laplace transform** $\hat{\Psi}$ of $\Phi = Z_l \vee Q_l$ satisfies the non-homogeneous ODE

$$\hat{\Psi}(s,x) = \int_0^\infty e^{-st}\Phi(t,x)\,dt\,, \qquad \left\{-\frac{\partial^2}{\partial x^2} + U(x) + s^2\right\}\hat{\Psi}(s,x) = -\mathcal{I}(s,x)\,, \tag{34}$$

where $\mathcal{I}(s,x) = -s\Psi(t,x)|_{t=0} - \partial_t\Psi(t,x)|_{t=0}$. The corresponding **homogeneous** equation is the **ODE we will study**:

$$\left\{-\frac{\partial^2}{\partial x^2} + U(x) + s^2\right\}\Psi(s,x) = 0\,. \tag{35}$$

- We consider **solutions** of (35) **regular** for $x \to \pm\infty$, that is infinity and horizon (with $\mathrm{Re}\, s > 0$)

$$\Psi_+(s,x) \sim e^{-sx}\,, \qquad x \to +\infty\,, \qquad \Psi_-(s,x) \sim e^{sx}\,, \qquad x \to -\infty\,. \tag{36}$$

- The Laplace tr. $\hat{\Psi}$ is given by the **Green function** G as (with $x_< = \min(x',x)$, $x_< = \max(x',x)$)

$$\hat{\Psi}(s,x) = \int_{-\infty}^\infty G(s,x,x')\mathcal{I}(s,x')dx'\,, \qquad G(s,x,x') = \frac{1}{W[\Psi_-,\Psi_+]}\Psi_-(s,x_<)\Psi_+(s,x_>)\,. \tag{37}$$

Quasinormal modes

- Finally taking the anti-Laplace transform of $\hat{\Psi}$ and setting $s = i\omega$ we get the **original perturbation** $\Phi = Z_l \vee Q_l$ as [Nollert, Class.Quant.Grav. 16 (1999)]

$$\Phi(t,x) = \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} e^{st}\hat{\Psi}(s,x)\, ds \tag{38}$$

$$= \sum_n e^{i\omega_n t}\mathrm{Res}\left(\frac{1}{W(s)}\right)\Bigg|_{\omega_n}\int_{-\infty}^{\infty}\Psi_-(\omega_n, x_<)\Psi_+(\omega_n, x_>)\mathcal{I}(\omega_n, x')\, dx' + (\text{other contributions})\,.$$

- The **quasinormal modes (QNMs)** are the complex frequencies (real frequencies and damping times) ω_n, which corresponds also to **zeros of the ODE Wronskian**

$$W(\omega_n) \equiv W[\Psi_-, \Psi_+](\omega_n) = 0\,, \tag{39}$$

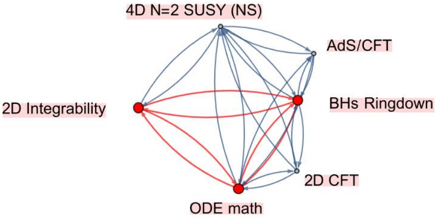
with $W[\Psi_-, \Psi_+] = \Psi_-(x)\Psi_+'(x) - \Psi_-'(x)\Psi_+(x)$.

Quasinormal modes of black holes as Bethe roots

- Through the ODE/IM formula $Q = W[\psi_+, \psi_-]$ we proved that the **Bethe root** (zero) condition on the Baxter's Q function

$$Q(\theta_n) = 0\,, \tag{40}$$

is equivalent to the mathematically rigorous definition of **quasinormal modes (QNMs)**! [Fioravanti-DG, arXiv:2112.11434]



- Q satisfies a QQ-**system** or quantum Wronskian **functional relation** which can be derived for **nearly any ODE**.
- In general the QQ-system **cannot** be directly inverted and solved as **integral equation**, rather it is necessary to construct **from** Q **the** Y **function** that solves the Y **system**.

Y function and quasinormal modes

- For example, for the **triad** [generalized Perturbed Hairpin/ $SU(2)$ $N_f = 2$ NS gauge th./ generalized extremal RN BHs] - *cf.* (12), (26) - one can define a Y **function** as [Fioravanti-DG, arXiv:2112.11434; Fioravanti-DG-Shu, arXiv:2208.14031]

$$Y_{+,+}(\theta) = e^{i\pi(q_1-q_2)} Q_{+,+}(\theta) Q_{-,-}(\theta) \,. \tag{41}$$

- In terms of Y the QQ **system** is written as

$$e^{i\pi(q_1+q_2)} Q_{+,+}(\theta + \frac{i\pi}{2}) Q_{-,-}(\theta - \frac{i\pi}{2}) = 1 + Y_{+,-}(\theta) \,. \tag{42}$$

and for **QNMs** follow the alternative **quantizations conditions**

$$Y_{+,-}(\theta_n - i\pi/2) = -1 \qquad Y_{-,+}(\theta_n + i\pi/2) = -1 \,. \tag{43}$$

- From the QQ system it follows the Y **system** as

$$Y_{+,-}(\theta + \frac{i\pi}{2}) Y_{-,+}(\theta - \frac{i\pi}{2}) = [1 + Y_{+,+}(\theta)][1 + Y_{-,-}(\theta)] \,. \tag{44}$$

Thermodynamic Bethe Ansatz

- Defining as the **pseudoenergy**

$$\varepsilon_{\pm,\pm}(\theta) = -\ln Y_{\pm,\pm}(\theta)\,, \tag{45}$$

- the Y **system can be inverted in the Thermodynamic Bethe Ansatz (TBA)** for it

$$\begin{aligned}
\varepsilon_{\pm,\pm}(\theta) &= \frac{8\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2} e^{\theta} \mp i\pi(q_1 - q_2) - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{\ln[1+\exp(-\varepsilon_{\pm,\mp}(\theta'))] + \ln[1+\exp(-\varepsilon_{\mp,\pm}(\theta'))]}{\cosh(\theta-\theta')} \\
\varepsilon_{\pm,\mp}(\theta) &= \frac{8\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2} e^{\theta} \mp i\pi(q_1 + q_2) - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{\ln[1+\exp(-\varepsilon_{\pm,\pm}(\theta'))] + \ln[1+\exp(-\varepsilon_{\mp,\mp}(\theta'))]}{\cosh(\theta-\theta')} -
\end{aligned} \tag{46}$$

- **Technicality**: P enters as the $\theta \to -\infty$ boundary condition $\varepsilon_{+,+}(\theta, P) \sim 4P\theta + 2C(P, q_1, q_2)$, as $\theta \to -\infty$ (following from the perturbative expansion of ODE).

Gravity Y system and TBA

- In **gravity variables** Σ_k (combinations of four charges for **generalised RN extremal BHs**) the Y system reads
[Fioravanti-DG, arXiv:2112.11434; Fioravanti-DG-Shu, arXiv:2208.14031]

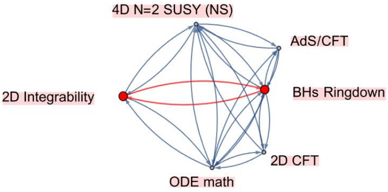
$$
\begin{aligned}
&Y(\theta+\frac{i\pi}{2},-i\Sigma_1,-\Sigma_2,i\Sigma_3,\Sigma_4)Y(\theta-\frac{i\pi}{2},-i\Sigma_1,-\Sigma_2,i\Sigma_3,\Sigma_4)\\
&=[1+Y(\theta,\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4)][1+Y(\theta,-\Sigma_1,\Sigma_2,-\Sigma_3,\Sigma_4)]
\end{aligned} \tag{47}
$$

- It can be inverted into the **TBA**

$$
\begin{aligned}
\varepsilon_\pm(\theta)&=[f_{0,+}\mp\frac{i\pi}{2}(\frac{\Sigma_1}{\Sigma_4^{1/4}}-\frac{\Sigma_3}{\Sigma_4^{3/4}})]e^\theta-\varphi*(\bar{L}_\pm+\bar{L}_\mp)(\theta)\\
\bar{\varepsilon}_\pm(\theta)&=[\bar{f}_{0,+}\pm\frac{\pi}{2}(\frac{\Sigma_1}{\Sigma_4^{1/4}}+\frac{\Sigma_3}{\Sigma_4^{3/4}})]e^\theta-\varphi*(L_\pm+L_\mp)(\theta)
\end{aligned} \tag{48}
$$

where we defined $\varepsilon_\pm(\theta)=-\ln Y(\theta,\pm\Sigma_1,\Sigma_2,\pm\Sigma_3,\Sigma_4)$, $\bar{\varepsilon}_\pm(\theta)=-\ln Y(\theta,\pm i\Sigma_1,-\Sigma_2,\mp i\Sigma_3,\Sigma_4)$, $L=\ln[1+\exp\{-\varepsilon\}]$, $\varphi(\theta)=(2\pi\cosh(\theta))^{-1}$...

Quasinormal modes from TBA



- The **QNMs condition** in gravity variables **reads alternatively as**

$$\bar{Y}_{+}(\theta_{n'} - i\pi/2) = -1\,, \quad Q_{+,+}(\theta_n) = 0\,, \qquad (49)$$
$$\bar{\varepsilon}_{+}(\theta_{n'} - i\pi/2) = -i\pi(2n'+1)\,.$$

- Through the quantization condition on $\bar{\varepsilon}_{+}$ we can **actually numerically compute QNMs from TBA**.

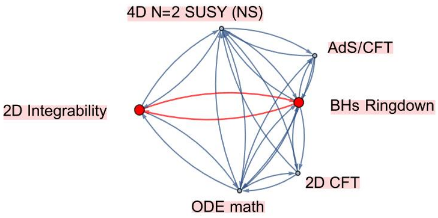
n	l	TBA	Leaver	WKB
0	1	0.869623 − 0.372022i	0.868932 − 0.372859i	0.89642 − 0.36596i
0	2	1.477990 − 0.368144i	1.477888 − 0.368240i	1.4940 − 0.36596i
0	3	2.080200 − 0.367076i	2.080168 − 0.367097i	2.0916 − 0.36596i
0	4	2.680363 − 0.366637i	2.680350 − 0.366642i	2.6893 − 0.36596i

Table: Comparison of QNMs in different methods ($n' = 0$, $\Sigma_1 = \Sigma_3 = 0.2$, $\Sigma_2 = 0.4$, $\Sigma_4 = 1$).

Other applications of integrability

- We notice that **much of the BH theory seems to go in parallel to the ODE/IM correspondence** construction and its 2D integrable field theory interpretation, beyond the determination of QNMs!
  - For instance, also the **greybody factor** that parametrizes the **Hawking radiation** seems to be **ratio of Qs**
  - Intepretation and use of T, TQ relation...

  [Fioravanti-DG, arXiv:2112.11434; Fioravanti-DG-Shu, arXiv:2208.14031; work in progress...]



Solving quantum Wronskian via $\mathcal{N} = 2$ SUSY

Solving QQ systems: D3 brane

- The scalar perturbation on the **D3 brane**, obeys **Mathieu equation** - (25) for $b=1$, slide 28 - a particular case of DCHE (12), which is the same as for self-dual Liouville IM and $N_f=0$ quantum SW curve.
- Through linear connection relations among solutions ψ_k one easily derives QQ-systems like

$$Q(\theta+i\pi/2,a)Q(\theta-i\pi/2,a)-Q(\theta,a)^2=1\,. \tag{50}$$

- From the QQ **system** follows a **closed formula for** Q

$$Q(\theta,a)=\frac{\sinh\left[\frac{1}{\hbar}A_D(\theta,a)\right]}{\sin\left[\frac{2\pi}{\hbar}a\right]}\,,\quad \text{with}\quad A_D\left(\theta+i\pi/2,a\right)=A_D(\theta,a)-2\pi ia\,, \tag{51}$$

We proved this and the following formulae purely from **ODE/IM and gauge theory**, but **alternative derivation** is possible through **AGT duality**

[Fioravanti-DG-Mahanta-Rossi,arXiv:2311.*****]

Solving QQ systems: extremal black holes

- We proved the QQ system for $N_f = 2$ gauge theory - *cf.* ODE (12), (26) - that is **extremal BHs**

$$Q\left(\theta+i\pi/2,-m_1,m_2\right)Q\left(\theta-i\pi/2,m_1,-m_2\right)-Q(\theta,-m_1,-m_2)Q(\theta,m_1,m_2)=e^{\frac{i\pi}{\hbar}(m_1-m_2)} \tag{52}$$

is **equivalent to the shift relations**

$$A_D\left(\theta\pm i\pi/2,m_1,-m_2,a\right)=A_D\left(\theta,m_1,m_2,a\right)+h\left(m_2,a\right)\mp i\pi a \tag{53}$$
$$(-\partial_{m_1}-\partial_{m_2})\mathcal{F}_{inst}(\theta+i\pi/2,-m_1,-m_2,a)+(\partial_{m_1}+\partial_{m_2})\mathcal{F}_{inst}(\theta-i\pi/2,m_1,m_2,a)=0$$

and similar others, with $h(m,a)=\ln\sqrt{\cos\frac{\pi}{\hbar}(a-m)\sec\frac{\pi}{\hbar}(a+m)}$.

- The **exact closed formula** for Q for $N_f = 2$ ensues

$$Q(\theta,m_1,m_2,a)=2\pi\exp\left\{\frac{1}{\hbar}\sum_{k=1}^{2}\frac{\partial\mathcal{F}\left(\theta,m_1,m_2,a\right)}{\partial m_k}\right\}\frac{\sinh\frac{1}{\hbar}A_D\left(\theta,m_1,m_2,a\right)}{\sin\frac{2\pi}{\hbar}a}. \tag{54}$$

Solving QQ systems: general (Minkowski) black holes (I)

- We proved the first QQ system for $N_f = 3$ gauge theory - *cf.* ODE (8), (11), (27) - that is **Schwarschild BHs**

$$Q_1(\theta + i\pi, m_1, m_2, -m_3)Q_1(\theta, -m_1, -m_2, m_3) - Q_1(\theta, m_1, m_2, m_3)Q_1(\theta + i\pi, -m_1, -m_2, -m_3) = \frac{2i}{\hbar}(m_1 + m_2)\,, \tag{55}$$

is **equivalent to the shift relations**

$$A_D(\theta \pm i\pi, m_1, m_2, -m_3, a) = A_D(\theta, m_1, m_2, m_3, a) + h(m_3, a) \mp i\pi a\,, \tag{56}$$

$$(\partial_{m_1} + \partial_{m_2} - \partial_{m_3})\mathcal{F}_{inst}(\theta + i\pi, m_1, m_2, -m_3, a) + (-\partial_{m_1} - \partial_{m_2} + \partial_{m_3})\mathcal{F}_{inst}(\theta, -m_1, -m_2, m_3, a) = 0$$

and similar others.

- The **exact closed formula** for Q_1 ensues

$$Q_1(\theta, \mathbf{m}, a) = \tag{57}$$

$$\pi i 2^{\frac{3}{2}} e^{\frac{1}{\hbar}\left[m_3\theta + \left(\frac{m_1+m_2}{2} + m_3 - \frac{\Lambda_3}{16}\right)\ln 2\right]}\Gamma\left(1 + \frac{m_1 + m_2}{\hbar}\right)\exp\left\{\frac{1}{\hbar}\sum_{k=1}^{3}\frac{\partial\mathcal{F}(\theta, \mathbf{m}, a)}{\partial m_k}\right\}\frac{\sinh\frac{1}{\hbar}A_D(\theta, \mathbf{m}, a)}{\sin\frac{2\pi}{\hbar}a}\,.$$

Solving QQ systems: general (Minkowski) black holes (II)

- We proved the second QQ system for $N_f=3$ gauge theory, that is **Schwarschild BHs**

$$Q_2(\theta,-q_2,-q_1,q_3)Q_2(\theta,q_2,q_1,q_3)-Q_2(\theta,q_1,q_2,q_3)Q_2(\theta,-q_1,-q_2,q_3)=q_1^2-q_2^2 \tag{58}$$

  is equivalent to the shift relation

$$\partial_{m_1}\mathcal{F}_{inst}(\theta,m_1,m_2,m_3,a)-\partial_{m_1}\mathcal{F}_{inst}(\theta,-m_1,-m_2,m_3,a)=0\,. \tag{59}$$

  and similar others

- The **exact closed formula** for Q_2 ensues

$$Q_2\left(\theta,\mathbf{m},a\right)=2^{\frac{m_2}{\hbar}}\Gamma(1+\frac{m_1+m_2}{\hbar})\Gamma(1+\frac{m_1-m_2}{\hbar})\exp\left\{\frac{2}{\hbar}\frac{\partial\mathcal{F}\left(\theta,\mathbf{m},a\right)}{\partial m_1}\right\}\,. \tag{60}$$

[Fioravanti-DG-Mahanta-Rossi,arXiv:2311.*****]

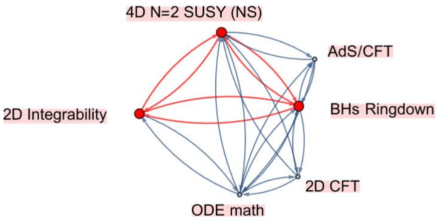
On the new ($\mathcal{N}=2$) gauge / gravity duality

- Since proved a relation between **at least one** Q **function and the gauge dual period**

$$Q(\theta, \mathbf{m}, a) = \mathfrak{F}(\theta, \mathbf{m}, a) \frac{\sinh \frac{1}{\hbar} A_D(\theta, \mathbf{m}, a)}{\sin \frac{2\pi}{\hbar} a}, \qquad (61)$$

  from our formalism **it follows immediately that QNMs are given by quantization conditions on the gauge periods** *cf.* (13)

$$Q(\theta, \mathbf{m}, a) = 0 \quad \Longrightarrow \quad A_D(\theta, \mathbf{m}, a) = 2i\pi\hbar\, n\,. \qquad (62)$$



- (62) was previously found heuristically and allowed:
  - a **novel analytic characterization of QNMs**;
  - to **find new results for the BHs theory** (by many other authors);
  - an **unexpected application of Supersymmetry**.

Comparison of QNMs computations methods

n	l	TBA	Leaver	WKB
0	1	$0.896681 - 0.40069i$	N.A.	$0.93069 - 0.39458i$
0	2	$1.5308 - 0.39676i$	N.A.	$1.5511 - 0.39458i$
0	3	$2.15708 - 0.395689i$	N.A.	$2.1716 - 0.39458i$
0	4	$2.78077 - 0.39525i$	N.A.	$2.7921 - 0.39458i$

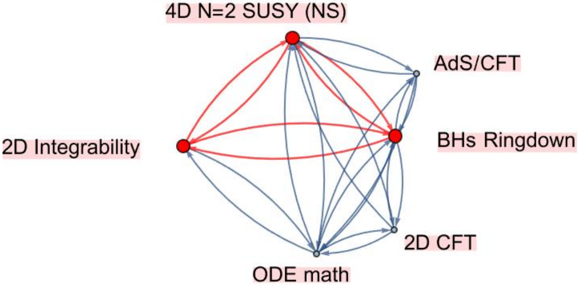
Table: Here $\Sigma_1 \neq \Sigma_3$ and the (original) Leaver method seems not applicable (N.A.).

- Computing QNMs exactly is a non-trivial mathematical problem, it has been typically quite **laborious**, also because of their **few exact analytic characterizations**.
- The standard analytic method is the one with the **continued fractions** by Leaver is laborious and we have found it **sometimes not applicable** (in its original form).
- Application of $\mathcal{N}=2$ **gauge theory** is a new analytic characterization of QNMs, but it requires a **nontrivial re-summation procedure** for $\omega_n \sim \Lambda_n \sim 1$.
- Our **integrability** exact method is **direct and simple**, but now we have developed for just a **few models**.

Conclusions

Summary

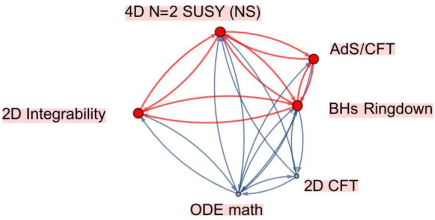
- We have shown **BHs perturbation theory** has a natural connection with
  - 4D $\mathcal{N} = 2$ **NS SUSY gauge theory**;
  - 2D **integrable models**;
  - many other related fields.
- In particular **ODE/IM correspondence** allowed
  - **understand the relation** among the three;
  - **find new results** an all sides, at the **exact level**.

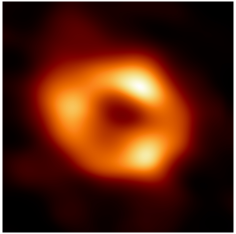


- Especially for QNMs of BHs we give:
  - **new analytic characterizations** from Q, Y, T functions;
  - **new method of computation** as Thermodynamic Bethe Ansatz.

Future developments

- $\mathcal{N} = 2$ gauge theories and BHs constitute **novel fields of application of integrability** and many future developments could be pursued:
  - generalization to **more BHs models**;
  - or **other BHs observables**;
  - **non-linear** perturbation theory;
  - **connection to other fields** (like holography).





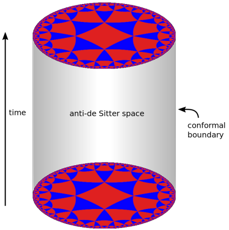
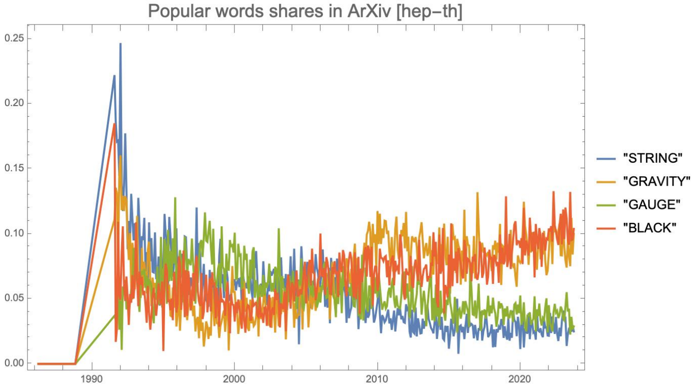


Figure: (Wikicommons/Herzog,Dunkel)

Thank you for your attention!



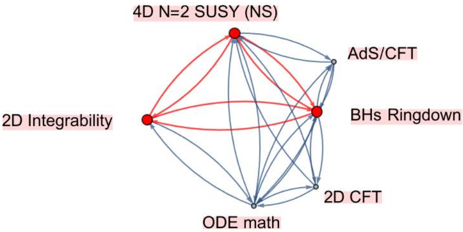


Figure: - GitHub

Figure: - Inspire